Essays on Sharing Economy

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Abstract

This thesis studies the product sharing manifestation of the sharing and on-demand economy. It consists of two essays, one on peer-to-peer (P2P) product sharing and the other on business-to-consumer (B2C) product sharing.

The first essay describes an equilibrium model of P2P sharing or collaborative consumption, where individuals with varying usage levels make decisions about whether or not to own a product. Owners are able to generate income from renting their products to non-owners while non-owners are able to access these products through renting on as needed basis. We characterize equilibrium outcomes, including ownership and usage levels, consumer surplus, and social welfare. We compare each outcome in systems with and without collaborative consumption. Our findings indicate that collaborative consumption can result in either lower or higher ownership and usage levels, with higher ownership and usage levels more likely when the cost of ownership is high. Our findings also indicate that consumers always benefit from collaborative consumption, with individuals who, in the absence of collaborative consumption, are indifferent between owning and not owning benefitting the most. We study both profit maximizing and social welfare maximizing platforms and compare equilibrium outcomes under both in terms of ownership, usage, and social welfare. We find that a not-for-profit platform would always charge a lower price and, therefore, lead to lower ownership and usage than a for-profit platform. We also examine the robustness of our results by considering several extensions to our model.

The second essay characterizes the optimal inventory repositioning policy for a class of B2C product sharing networks. We consider a B2C product sharing network with a fixed number of rental units distributed across multiple locations. The units are accessed by customers without prior reservation and on an on-demand basis. Customers are provided with the flexibility to decide on how long to keep a unit and where to return it. Because of the randomness in demand, rental periods and return locations, there is a need to periodically reposition inventory away from some locations and into others. In deciding on how much inventory to reposition and where, the system manager balances potential lost sales with repositioning costs. We formulate the problem into a Markov decision process and show that the problem in each period is one that involves solving a convex optimization problem. The optimal policy in each period can be described in terms of a well-specified region over the state space. Within this region, it is optimal not to reposition any inventory while, outside the region, it is optimal to reposition some inventory but only such that the system moves to a new state that is on the boundary of the no-repositioning region. We provide a simple check for when a state is in the no-repositioning region, which also allows us to compute the optimal policy more efficiently.

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Chapter 1

Introduction

We are witnessing, across a wide range of domains, a shift away from the exclusive ownership and planned consumption of resources to one of shared use and on-demand consumption. This shift is taking advantage of technological development such as online marketplaces and mediation platforms. Value is derived from the fact that many resources are acquired to satisfy infrequent demand but are otherwise poorly utilized (for example, the average car in the US is used less than 5 percent of the time). Several successful businesses in the US and elsewhere, such as Turo for cars, Nice Ride for bikes, Style Lend for designer clothing, 3D Hubs for 3D printers, Uber for transportation services, TaskRabbit for errands and Postmates for courier services, provide a proof of concept and evidence of viability for the shared and on-demand access to products and services. Collectively, these businesses and others are giving rise to what is becoming known as the sharing and on-demand economy.

The term sharing economy refers to the economic activities in which participants share the access to products and services through online market places. It encompasses product sharing platforms such as Turo, 3D Hubs and Nice Ride, as well as service sharing platforms such as Uber, TaskRabbit and Postmates.[1] A common feature of these platforms is the ability to fulfill consumer demand via the immediate provisioning of products and services. Economic activities with this feature are referred to as the on-demand economy. In this thesis, we focus on the product sharing aspect of the sharing and on-demand economy.

Product sharing is not a new concept. However, recent technological advances in several areas have made it more feasible by lowering the associated search and transaction costs. These advances include the development of online marketplaces, mobile devices and platforms, electronic payments, and two-way reputation systems whereby users rate providers and providers rate users. Other drivers behind the rise of product sharing are societal and include increased population density in urban areas around the world, increased concern about the environment (product sharing is viewed as a more sustainable alternative to traditional modes of consumption), increased desire for convenience, and increased desire for community and altruism among the young and educated.

Product sharing can in general be classified into two catergories, peer-to-peer (P2P) product sharing and business-to-consumer (B2C) product sharing. P2P product sharing enables the peer-to-peer rental of products, whereby owners rent

[1]Service sharing platforms often employ self-scheduled workers, and focus on how the supply and demand of services are affected by the prices and wages, and how the choices of these prices and wages affect customer delay and platform profit. For a discussion of these platforms, see Gurvich et al. [2015]; Taylor [2016]; Cachon et al. [2015]; Banerjee et al. [2015] and the references therein.

on a short-term basis poorly utilized assets to non-owners and non-owners access these assets through renting on an as-needed basis. The platform in P2P product sharing does not own the products, but serves to facilitate the transactions between owners and renters. P2P product sharing has the potential of increasing access while reducing investments in resources and infrastructure. In turn, this could have the twin benefit of improving consumer welfare (individuals who may not otherwise afford a product now have an opportunity to use it) while reducing societal costs (externalities, such as pollution that may be associated with the production, distribution, use, and disposal of the product). Take cars for example. The availability of sharing could lead some to forego car ownership in favor of on-demand access. In turn, this could result in a corresponding reduction in congestion and emissions and, eventually, in reduced investments in roads and parking infrastructure. However, increased sharing may have other consequences, some of which may be undesirable. For example, greater access to cars could increase car usage and, therefore, lead to more congestion and pollution if it is not accompanied by a sufficient reduction in the number of cars.[2]

P2P product sharing raises several important questions. How does P2P product sharing affect ownership and usage of products? Is it necessarily the case that P2P product sharing leads to lower ownership, lower usage, or both (and therefore to improved environmental impact)? If not, what conditions would favor lower

[2]An article in the New York Times (Board [2015]) notes that "The average daytime speed of cars in Manhattan's business districts has fallen to just under 8 miles per hour this year, from about 9.15 miles per hour in 2009. City officials say that car services like Uber and Lyft are partly to blame. So Mayor Bill de Blasio is proposing to cap their growth." A recent study by KPMG (Korosec [2015]) projects a significant increase in miles driven by cars due to increased usage of on-demand transportation services by two ends of the population demographic spectrum, the very young (children and teenagers) and the old.

ownership, lower usage, or both? Who benefits the most from collaborative consumption among owners and renters? To what extent would a profit maximizing platform, through its choice of rental prices, improve social welfare? To what extent do frictions, such as extra wear and tear renters place on rented products and inconvenience experienced by owners affect platform profit and social welfare?

In Chapter 2, we address these and other related questions. We describe an equilibrium model of peer-to-peer product sharing, where individuals with varying usage levels make decisions about whether or not to own. In the presence of collaborative consumption, owners are able to generate income from renting their products to non-owners while non-owners are able to access these products through renting. The matching of owners and renters is facilitated by a platform, which sets the rental price and charges a commission fee. We characterize equilibrium outcomes, including ownership and usage levels, consumer surplus, and social welfare. Our findings indicate that, depending on the rental price, collaborative consumption can result in either lower or higher ownership and usage levels, with higher ownership and usage levels more likely when the cost of ownership is high. Our findings also indicate that consumers always benefit from collaborative consumption, with individuals who, in the absence of collaborative consumption, are indifferent between owning and not owning benefitting the most. We show that these results continue to hold in settings where the rental price is determined by either a for-profit (profit maximizing) or a not-for-profit (social welfare maximizing) platform. We find that a not-for-profit platform would always charge a lower price and, therefore, lead to lower ownership and usage than a for-profit platform. We also examine the robustness of our results by considering several extensions to our model.

B2C product sharing, on the other hand, is on-demand product rental in disguise. Like other on-demand businesses, it improves upon traditional product rental by providing consumers with cashless and cashier-less service experience and 24 by 7 access to products with more flexible rental periods at more convenient locations. The most innovative B2C sharing businesses allow their customers to pick up a product without reservation and, to keep the product for one or more periods without committing to a specific return time or location. Systems with these features include Redbox for disc rental, Nice Ride for bike sharing and Zipcar for car sharing (Point-to-point car sharing programs such as Car2go can also be treated as location based, with each location being either a neighborhood, a city block, or a public parking lot). When operated at a high service level, B2C product sharing provides a level of freedom that is highly sought after by consumers.[3] However, due to the constant flow of products between locations, such a system can become quickly imbalanced and, therefore, requires periodic inventory repositioning to effectively match supply and demand.

Although the problem is common in practice, and carries significant economic costs for the affected firms and their customers, the existing literature on this topic is relatively limited. In particular, how to manage these systems optimally

[3]Due to these value-added features, product sharing has been well received by both consumers and service providers. Take car sharing for example, the global car sharing members increased from $346,610$ to $4,842,616$ from 2006 to 2014 (a 40% annual growth rate), and the corresponding fleet size grew from $11,501$ to $104,125$ (a 32% annual growth rate). See Shaheen and Cohen [2016]. By 2015, Hertz, Avis Budget and Enterprise, who control 95% of the car rental market in North America and a sizeable market elsewhere, all have acquired their own car sharing businesses (Zipcar by Avis Budget, Enterprise CarShare by Enterprise, and Hertz 24/7 by Hertz).

is, to the best of our knowledge, not known. Moreover, there does not appear to be efficient methods for computing the optimal policy for systems as general as the one we consider in this research, including effective heuristics. This relative lack of results appears to be due to the multidimensional nature of the problem as well as the lost sales feature, compounded by the presence of randomness in demand, rental periods and return locations.

In Chapter 3, we address some of these limitations. We focus on the Redbox model, where rental periods are longer than review periods. We formulate the problem as a Markov decision process and show that the problem to be solved in each period is one that involves solving a convex optimization problem. More significantly, we show that the optimal policy in each period can be described in terms of two well-specified regions over the state space. If the system is in a state that falls within one region, it is optimal not to reposition any inventory (we refer to this region as the "no-repositioning" region). If the system is in a state that is outside this region, it is optimal to reposition some inventory but only such that the system moves to a new state that is on the boundary of the no-repositioning region. Moreover, we provide a simple check for when a state is in the no-repositioning region, which also allows us to compute the optimal policy more efficiently.

The rest of the thesis is organized as follows. In Chapter 2, we provide treatment for P2P product sharing. In chapter 3, we address the inventory repositioning problem in B2C product sharing. In Chapter 4, we offer concluding comments and discuss plans for future research.

Chapter 2

Peer-to-Peer Product Sharing: Implications for Ownership, Usage and Social Welfare

2.1 Introduction

In this chapter, we describe an equilibrium model of peer-to-peer product sharing or collaborative consumption, where individuals with varying usage levels make decisions about whether or not to own a homogenous product. In the presence of collaborative consumption, owners are able to generate income from renting their products to non-owners while non-owners are able to access these products through renting. The matching of owners and renters is facilitated by a platform, which sets the rental price and charges a commission fee.[1] Because supply and demand can fluctuate over the short run, we allow for the possibility that an owner may not always be able to find a renter when she puts her product up for rent. Similarly, we allow for the possibility that a renter may not always be able to find a product to rent when he needs one. We refer to the uncertainty regarding the availability of renters and products as matching friction and describe a model for this uncertainty. We also account for the cost incurred by owners due to the extra wear and tear that a renter places on a rented product and for the inconvenience cost experienced by renters for using a product that is not their own.

For a given price and a commission rate, we characterize equilibrium ownership and usage levels, consumer surplus, and social welfare. We compare each in systems with and without collaborative consumption and examine the impact of various problem parameters including price, commission rate, cost of ownership, extra wear and tear cost, and inconvenience cost. We also do so when the price is a decision made by the platform to maximize either profit or social welfare. Our main findings include the following:

- Depending on the rental price, we show that collaborative consumption can result in either higher or lower ownership. In particular, we show that when the rental price is sufficiently high (above a well-specified threshold), collaborative consumption leads to higher ownership. We show that this threshold

[1] A variety of pricing approaches are observed in practice. Some platforms allow owners to choose their own prices. Others (e.g., DriveMycar) determine the price. There are also cases where the approach is hybrid, with owners determining a minimum acceptable price but allowing the platform to adjust it higher (e.g., Turo), or with the platform suggesting a price (e.g., JustShareIt) but allowing owners to deviate. From conversations the authors had with several industry executives, there appears to be a push toward platform pricing, with several platforms investing in the development of sophisticated pricing engines to support owners.

is decreasing in the cost of ownership. That is, collaborative consumption is more likely to lead to more ownership when the cost of ownership is high (this is because collaborative consumption allows individuals to offset the high ownership cost and pulls in a segment of the population that may not otherwise choose to own).

- Similarly, we show that collaborative consumption can lead to either higher or lower usage, with usage being higher when price is sufficiently high. Thus, it is possible for collaborative consumption to result in both higher ownership and higher usage (it is also possible for ownership to be lower but usage to be higher and for both ownership and usage to be lower).

- These results continue to hold in settings where the rental price is determined by a profit maximizing or a social welfare maximizing platform. In particular, collaborative consumption can still lead to either higher or lower ownership and usage with higher ownership and usage more likely when the cost of ownership is higher.

- We show that consumers always benefit from collaborative consumption, with individuals who, in the absence of collaborative consumption, are indifferent between owning and not owning benefitting the most. This is because among non-owners those with the most usage (and therefore end up renting the most) benefit the most from collaborative consumption. Similarly, among owners, those with the least usage (and therefore end up earning the most rental income) benefit the most.

- For a profit maximizing platform, we show that profit is not monotonic in the cost of ownership, implying that a platform is least profitable when the cost of ownership is either very high or very low (those two extremes lead to scenarios with either mostly renters and few owners or mostly owners and few renters). The platform is most profitable when owners and renters are sufficiently balanced. For similar reasons, social welfare is also highest when owners and renters are sufficiently balanced.

- We observe that platform profit is also not monotonic in the extra wear and tear renters place on a rented product, implying that a platform may not always have an incentive to reduce this cost. This is because the platform can leverage this cost to induce desirable ownership levels without resorting to extreme pricing, which can be detrimental to its revenue.

- We examine the robustness of our results by considering settings, among others, where (1) non-owners have the option of renting from a third party service provider, (2) platforms may own assets of their own, (3) individuals are heterogeneous in their aversion to renting to/from others (i.e., their sensitivity to the costs of extra wear and tear and inconvenience), (4) usage is endogenous, and (5) usage has a general distribution.

The rest of the chapter is organized as follows. In Section 2.2, we provide a review of related literature. In Section 2.3, we describe our model. In Section 2.4, we provide an analysis of the equilibrium. In Section 2.5, we consider the platform's problem. In Section 2.6, we discuss extensions.

2.2 Related Literature

Our work is related to the literature on two-sided markets (see for example Rochet and Tirole [2006]; Weyl [2010]; and Hagiu and Wright [2015]). Examples of two-sided markets include social media platforms which bring together members and advertisers or operating systems for computers and smart phones, which connect users and application developers. A common feature of two-sided markets is that the utility of individuals on each side of the market increases with the size of the other side of the market. As a result, it can be beneficial for the platform to heavily subsidize one side of the market (e.g., social media sites are typically free to members). Collaborative consumption is different from two-sided markets in several ways, the most important of which is that the two sides are not distinct. In collaborative consumption, being either an owner or a renter is a decision that users of the platform make, with more owners implying fewer renters, and vice-versa. Therefore, heavily subsidizing one side of the market may not necessarily be desirable as it can create an imbalance in the supply and demand for the shared resource.

Our work is also related to the literature on *servicization*. Servicization refers to a business model under which a firm that supplies a product to the market retains ownership of the product and instead charges customers per use (e.g., printer manufacturers charging customers per printed page instead of charging them for the purchase of a printer or car manufacturers renting cars on a short term basis instead of selling them or leasing them on a long term basis). Agrawal and Bellos [2016] examine the extent to which servicization affects ownership and usage and the associated environmental impact.[2] Orsdemir et al. [2017] evaluate both the profitability and the environmental impact of servicization. Bellos et al. [2013] study the economic and environmental implications of an auto manufacturer, in addition to selling cars, offering a car sharing service. Additional discussion and examples of servicization can be found in Agrawal and Bellos [2016] and the references therein. Peer-to-peer product sharing is different from servicization in that there is no single entity that owns the rental units, with owners being simultaneously consumers and suppliers of the market. As a result, the payoff of one side of the market depends on the availability of the other side. This, coupled with the fact that supply and demand are not guaranteed to be matched with each other, makes ownership and usage decisions more complicated than those under servicization.

There is a growing body of literature on peer-to-peer markets (see Einav et al. [2016] for a recent review). Within this literature, there is a small but growing stream that deals with peer-to-peer markets with collaborative consumption features. Fradkin et al. [2015] studies sources of inefficiency in matching buyers and suppliers in online market places. Using a counterfactual study, they show how changes to the ranking algorithm of Airbnb can improve the rate at which buyers are successfully matched with suppliers. Zervas et al. [2015] examine the relationship between Airbnb supply and hotel room revenue and find that an increase

[2]Under a servicization model, the firm can exert costly effort to improve certain characteristics of the product such as its energy efficiency during use or its durability. This could lower the corresponding operating costs, which in turn could result in higher usage. The phenomenon of higher efficiency leading to more usage is commonly referred to as the *rebound effect*. See Greening et al. [2000] for an overview and references. In our setting, the introduction of collaborative consumption can lead, under some conditions, to higher ownership because of the rental income owners derive from ownership.

in Airbnb supply has only a modest negative impact on hotel revenue. Cullen and Farronato [2014] describe a model of peer-to-peer labor marketplaces. They calibrate the model using data from TaskRabbit and find that supply is highly elastic, with increases in demand matched by increases in supply per worker with little or no impact on price.

Papers that are closest in spirit to ours are Fraiberger and Sundararajan [2016] and Jiang and Tian [2016]. Fraiberger and Sundararajan [2016] describe a dynamic programing model where individuals make decisions in each period regarding whether to purchase a new car, purchase a used a car, or not purchase anything. They model matching friction, as we do, but assume that the renter-owner matching probabilities are exogenously specified and not affected by the ratio of owners to renters (in our case, we allow for these to depend on the ratio of owners to renters which turns out to be critical in the decisions of individuals on whether to own or rent). They use the model to carry out a numerical study. For the parameter values they consider, they show that collaborative consumption leads to a reduction in new and used car ownership, an increase in the fraction of the population who do not own, and an increase in the usage intensity per vehicle. In this paper, we show that ownership and usage can actually either increase or decrease with collaborative consumption and provide analytical results regarding conditions under which different combinations of outcomes can occur. We also study the decision of the platform regarding pricing and the impact of various parameters on platform profitability.

Jiang and Tian [2016] describe a two-period model, where individuals first decide on whether or not to own a product. This is followed by owners deciding in each period on whether to use the product themselves or rent it. They assume that demand always matches supply through a market clearing price and do not consider, as we do, the possibility of a mismatch, because of matching friction, between supply and demand. They focus on the decision of the product manufacturer. In particular, they study how the manufacturer should choose its retail price and product quality in anticipation of sharing by consumers. In contrast, we focus on the decision of the platform which in our case decides on the rental price.

Empirical studies that examine the impact of peer-to-peer product sharing on ownership and usage are scarce. Clark et al. [2014] present results from a survey of British users of a peer-to-peer car sharing service. They find that peer-to-peer car sharing has led to a net increase in the number of miles driven by car renters. Linden and Franciscus [2016] examine differences in the prevalence of peer-to-peer car sharing among several European cities. He finds that peer-to-peer car sharing is more prevalent in cities where a larger share of trips is taken by public transport and where there is a city center less suitable for car use. Ballus-Armet et al. [2014] report on a survey in San Francisco of public perception of peer-to-peer car sharing. They find that approximately 25% of surveyed car owners would be willing to share their personal vehicles through peer-to-peer car sharing, with liability and trust concerns being the primary deterrents. They also find that those who drive almost every day are less likely to rent through peer-to-peer car sharing, while those who use public transit at least once per week are more likely to do so. There are a few studies that consider car sharing that involves a third party service provider, such as a car rental company. For example, Nijland et al.

[2015] (and also Martin and Shaheen [2011]) find that car sharing would lead to a net decrease in car usage. On the other hand, a study by KPMG (Korosec [2015]) projects a significant increase in miles driven by cars and attributes this to increased usage of on-demand transportation services. In general, there does not appear to be a consensus yet on the impact of car sharing on car usage and ownership. Our paper, by providing a framework for understanding how various factors may affect product sharing outcomes, could be useful in informing future empirical studies.

2.3 Model Description

In this section, we describe our model of collaborative consumption. The model is applicable to the case of peer to peer product sharing where owners make their products available for rent when they are not using them and non-owners can rent from owners to fulfill their usage needs. We reference the case of car sharing. However, the model applies more broadly to the collaborative consumption of other products. We consider a population of individuals who are heterogeneous in their product usage, with their type characterized by their usage level ξ. We assume usage is exogenously determined (i.e., the usage of each individual is mostly inflexible). In Section 2.6.4, we consider the case where usage is endogenously determined and affected by the presence of collaborative consumption. We assume that the utility derived by an individual with type ξ, $u(\xi)$ is linear in ξ with $u(\xi) = \xi$. We use a linear utility for ease of exposition and to allow for closed form expressions. A linear utility has constant returns to scale, and, without loss of generality, the utility derived from each unit of usage can be normalized to 1. Also without loss of generality, we normalize the usage level to $[0, 1]$, where $\xi = 0$ corresponds to no usage at all and $\xi = 1$ to full usage. We let $f(\xi)$ denote the density function of the usage distribution in the population.

We assume products are homogeneous in their features, quality, and cost of ownership. In the absence of collaborative consumption, each individual makes a decision about whether or not to own. In the presence of collaborative consumption, each individual decides on whether to own, rent from others who own, or neither. Owners incur the fixed cost of ownership but can now generate income by renting their products to others who choose not to own. Renters pay the rental fee but avoid the fixed cost of ownership.

We let p denote the rental price per unit of usage that renters pay (a uniform price is consistent with observed practices by certain peer-to-peer platforms when the goods are homogenous). This rental price may be set by a third party platform (an entity that may be motivated by profit, social welfare, or some other concern; see Section 5 for further discussion). The platform extracts a commission from successful transactions, which we denote by γ, where $0 \leq \gamma < 1$, so that the rental income seen by the owner per unit of usage is $(1 - \gamma)p$. We let α, where $0 \leq \alpha \leq 1$ denote the fraction of time in equilibrium that an owner, whenever she puts her product up for rent, is successful in finding a renter. Similarly, we denote by β, where $0 \leq \beta \leq 1$, fraction of time that a renter, whenever he decides to rent, is successful in finding an available product (the parameters α and β are determined endogenously in equilibrium). A renter resorts to his outside option (e.g., public transport in the case of cars) whenever he is not successful in finding a product to rent. The owner incurs a fixed cost of ownership, denoted by c, which may include not just the purchase cost (if costs are expressed per unit time, this cost would be amortized accordingly) but also other ownership-related costs such as those related to storage and insurance. Whenever the product is rented, the owner incurs an additional cost, denoted by d_o, due to extra wear and tear the renter places on the product. Renters, on the other hand, incur an inconvenience cost, denoted by d_r (in addition to paying the rental fee), from using someone else's product and not their own. Without loss of generality, we assume that $c, p, d_o, d_r \in [0, 1]$ and normalize the value of the outside option (e.g., using public transport) to 0.

We assume that $p(1 - \gamma) \geq d_o$ so that an owner would always put her product out for rent when she is not using it. Note that usage corresponds to the portion of time an owner would like to have access to her product, regardless of whether or not she is actually using it. An owner has always priority in accessing her product. Hence her usage can always be fulfilled. We also assume that $p + d_r \leq 1$ so that a renter always prefers renting to the outside option. Otherwise, rentals would never take place as the outside option is assumed to be always available. There are of course settings where an individual would like to use a mix of options (e.g., different transportation methods). In that case, ξ corresponds to the portion of usage that an individual prefers to fulfill using the product (e.g., a car and not public transport).

The *payoff* of an owner with usage level ξ can now be expressed as

$$\pi_o(\xi) = \xi + (1 - \xi)\alpha[(1 - \gamma)p - d_o] - c, \tag{2.1}$$

while the payoff of a renter as

$$\pi_r(\xi) = \beta\xi - \beta(p + d_r)\xi. \tag{2.2}$$

The payoff of an owner has three terms: the utility derived from usage, the income derived from renting (net of the wear and tear cost), and the cost of ownership. The income from renting is realized only when the owner is able to find a renter. The payoff of a renter is the difference between the utility derived from renting and the cost of renting (the sum of rental price and inconvenience cost). A renter derives utility and incurs costs whenever he is successful in renting a product.

An individual with type ξ would participate in collaborative consumption as an *owner* if the following conditions are satisfied

$$\pi_o(\xi) \geq \pi_r(\xi) \text{ and } \pi_o(\xi) \geq 0.$$

The first constraint ensures that an individual who chooses to be an owner prefers to be an owner to being a renter. The second constraint is a participation constraint that ensures the individual participates in collaborative consumption. Similarly, an individual with type ξ would participate in collaborative consumption as a *renter* if the following conditions are satisfied

$$\pi_r(\xi) \geq \pi_o(\xi) \text{ and } \pi_r(\xi) \geq 0.$$

Noting that, for any given pair of α and β in $[0, 1]$, $\pi_o(\xi) - \pi_r(\xi)$ is monotonically increasing and $\pi_r(\xi) \geq 0$ for $\xi \in [0, 1]$, collaborative consumption would take place if there exists $\theta \in (0, 1)$ such that

$$\pi_o(\theta) = \pi_r(\theta). \tag{2.3}$$

The parameter θ would then segment the population into owners and renters, where individuals with $\xi > \theta$ are owners and individuals with $\xi < \theta$ are renters (an individual with $\xi = \theta$ is indifferent between owning and renting). We refer to

$$\omega = \int_{[\theta,1]} f(\xi)\, d\xi,$$

the fraction of owners in the population, as the *ownership level* or simply *ownership*. In addition, we refer to

$$q(\theta) = \int_{[\theta,1]} \xi f(\xi)\, d\xi + \beta \int_{[0,\theta]} \xi f(\xi)\, d\xi,$$

the total usage generated from the population, as the *usage level* or simply *usage*. Note that the first term is usage due to owners, and the second term is usage due to renters (and hence modulated by β).

2.3.1 Matching Supply with Demand

In the presence of collaborative consumption, let $D(\theta)$ denote the aggregate demand (for rentals) generated by renters and $S(\theta)$ the aggregate supply generated by owners, for given θ. Then,

$$D(\theta) = \int_{[0,\theta)} \xi f(\xi)\, d\xi$$

and

$$S(\theta) = \int_{[\theta,1]} (1-\xi) f(\xi)\, d\xi.$$

Moreover, the amount of demand from renters that is fulfilled must equal the amount of supply from owners that is matched with renters. In other words, the following fundamental relationship must be satisfied

$$\alpha S(\theta) = \beta D(\theta). \tag{2.4}$$

The parameters α and β, along with θ, are determined endogenously in equilibrium.

As mentioned earlier, matching friction can arise because of short term fluctuations in supply and demand (even though overall supply and demand are constant in the long run). This short term fluctuation may be due to the inherent variability in the timing of individual rental requests or in the duration of individual rental periods. Consequently, an available product may not find an immediate renter and a renter may not always be able to find an available product. In constructing a model for α and β, the following are desirable properties: (*i*) α (β) increases (deceases) in θ; (*ii*) α approaches 1 (0) when θ approaches 1 (0); (*iii*) β approaches 1 (0) when θ approaches 0 (1), and (*iv*) α and β must satisfy the supply-demand relationship in (2.4).

Below we describe a plausible model for the short term dynamics of matching owners and renters. This model is by no means unique and in Appendix 2.C we describe an alternative approach to model these dynamics. The model takes the view that in the short term (e.g., over the course of a day) demand is not realized all at once but requests for rentals arise continuously over time with random interarrival times. The intensity of the arrival process is of course determined by the total demand (e.g., total demand per day). The supply translates into individual products available for rent (for simplicity assume that supply is realized all at once and does not fluctuate over the time during which rental requests arrive). Once a product is rented, it becomes unavailable for the duration of the rental time, which may also be random. Because of the randomness in the interarrival times between requests and rental times per request, a request may arrive and find all products rented out. Assuming renters do not wait for a product to become available, such a request would then go unfulfilled. Also, because of this randomness, a product may not be rented all the time even if total demand exceeds total supply.

The dynamics described above are similar to those of a *multi-server loss queueing system*[3]. In such a system, $1-\beta$ would correspond to the *blocking* probability (the probability that a rental request finds all products rented out, or, in queueing parlance, the arrival of a request finds all servers busy) while α would correspond to the utilization of the servers (the probability that a product is being rented out).

If we let m denote the mean rental time per rental, the arrival rate (in terms of rental requests per unit time) is given by $\lambda(\theta) = \frac{D(\theta)}{m}$, and service capacity

[3]In a multi-server loss queueing system, customers arrive over time to receive service from a set of identical servers. A customer who does not find an available server upon arrival leaves the system without getting service. A customer who finds one or more available servers proceeds to receive service from one of these servers. Service takes a specified amount of time. Upon completion of service, the corresponding server becomes available. Both the interarrival and service times can be stochastic. (See Cooper [1981] for additional details).

(the number of rental requests that can be fulfilled per unit time) by $\mu(\theta) = \frac{S(\theta)}{m}$.[4] Therefore, we can express the workload (the ratio of the arrival rate to the service capacity) of the system as $\rho(\theta) = \frac{\lambda(\theta)}{\mu(\theta)} = \frac{D(\theta)}{S(\theta)}$ and the utilization as $\alpha = \frac{\beta\lambda(\theta)}{\mu(\theta)} = \frac{\beta D(\theta)}{S(\theta)}$ (these relationships are of course consistent with the supply-demand relationship in (2.4)).

Let k denote the number of servers (as it turns out, we do not have to compute k explicitly as the approximation we end up using to estimate α and β depends only on the workload $\rho(\theta)$). Then, assuming we can approximate the arrival process by a Poisson process, the blocking probability $B(\rho, k)(= 1 - \beta)$ is given by the *Erlang loss formula* (see for example Sevastyanov [1957])

$$B(\rho, k) = \frac{\frac{(k\rho)^k}{k!}}{\sum_{n=0}^{k} \frac{(k\rho)^n}{n!}}.$$

Unfortunately the above expression is not easily amenable to mathematical analysis. Therefore, in what follows we consider approximations that are more tractable yet retain the desirable properties (i)-(iv).

Sobel [1980] provides the following lower and upper bounds for the blocking probability:

$$(1 - \frac{1}{\rho})^+ \leq B(\rho, k) \leq 1 - \frac{1}{1 + \rho}.$$

Both the lower and upper bounds arise from approximations. The lower bound is obtained from a *deterministic fluid* approximation where $B(\rho, k) = 0$ if $D(\theta) \leq$

[4]For example, suppose the aggregate demand for renting per unit time is $D(\theta) = 1000$ hours and the aggregate supply for renting per unit time is $S(\theta) = 2000$ hours. If the average rental period is $m = 5$ hours, then the arrival rate and the service capacity of the system are respectively $\lambda(\theta) = D(\theta)/m = 200$ and $\mu(\theta) = S(\theta)/m = 400$ requests per unit time.

$S(\theta)$ and $B(\rho, k) = 1 - S(\theta)/D(\theta)$ otherwise. This leads to the following approximation of β and α

$$\beta = \min\{\frac{1}{\rho}, 1\} = \min\{\frac{S(\theta)}{D(\theta)}, 1\},$$

and

$$\alpha = \min\{\rho, 1\} = \min\{\frac{D(\theta)}{S(\theta)}, 1\}.$$

Clearly, these approximations are upper bounds on the exact matching probabilities and therefore overestimate these probabilities (this is not surprising given that these matching probabilities are those of a deterministic system). Note also that at least one of these expressions is one, so that $\alpha + \beta > 1$. This is perhaps reasonable when short term fluctuations in supply and demand are insignificant and friction arises only from the fact that, in equilibrium, either supply can exceed demand or vice-versa. However, these approximations are in contrast with the empirical evidence that α and β can be simultaneously less than one.[5]

The upper bound arises from approximating the multi-server system by a single server system with an equivalent service capacity. This leads to the following approximation of β and α

$$\beta = \frac{1}{1+\rho} = \frac{S(\theta)}{S(\theta) + D(\theta)}, \tag{2.5}$$

and

$$\alpha = \frac{\rho}{1+\rho} = \frac{D(\theta)}{S(\theta) + D(\theta)}. \tag{2.6}$$

These approximations are lower bounds on the the exact matching probabilities

[5]Using data from Getaround, Fraiberger and Sundararajan [2016] estimate the matching probabilities, for the case study they consider, to be $\alpha = 0.1$ and $\beta = 0.6$ (self reported values based on a survey of Getaround users put the estimated β at a slightly higher value of 0.85).

and therefore underestimate these probabilities. Note that in this case $\alpha + \beta = 1$. Interestingly, these expressions can be obtained directly from the supply-demand relationship in (2.4) if we require that $\alpha + \beta = 1$ (the above expressions are in that case the unique solution to (2.4)).

Both of the upper and lower bound approximations satisfy properties (i)-(iv) described above. However, the approximations in (2.5) and (2.6) allow for both α and β to be strictly less than one and for the possibility of matching friction for both owners and renters. Therefore, in the rest of the chapter, we rely on this approximation for our analysis. We are nonetheless able to confirm that all the results we obtain are qualitatively the same as those obtained under the upper bound approximation.

We are now ready to proceed with the analysis of the equilibrium. An equilibrium under collaborative consumption exists if there exists $(\theta, \alpha) \in (0, 1)^2$ that is a solution to (2.3) and (2.6). When it exists, we denote this solution by (θ^*, α^*). Knowing the equilibrium allows us to answer important questions regarding product ownership, usage, and social welfare, among others.

2.4 Equilibrium Analysis

In this section, we consider the case where the price is exogenously specified. In Section 2.5, we treat the case where the price is chosen optimally by the platform. As mentioned in Section 2.3, the rental price must satisfy $\frac{d_o}{1-\gamma} \leq p \leq 1 - d_r$, since otherwise, either the owners or renters will not participate. We denote the set of admissible prices by $A = [\frac{d_o}{1-\gamma}, 1 - d_r]$. For ease of exposition and to allow for closed form expressions, we assume that ξ is uniformly distributed in $[0,1]$ (we consider more general distributions in Section 2.6.5).

Letting θ denote the solution to $\pi_o(\xi) = \pi_r(\xi)$ leads to

$$\theta = \frac{c - ((1-\gamma)p - d_o)\alpha}{p + d_r + (1 - p - d_r)\alpha - ((1-\gamma)p - d_o)\alpha}. \tag{2.7}$$

Given θ, the aggregate demand under collaborative consumption is given by $D(\theta) = \frac{\theta^2}{2}$ and the aggregate supply by $S(\theta) = \frac{(1-\theta)^2}{2}$. This leads to $\rho(\theta) = \frac{\theta^2}{(1-\theta)^2}$, and by (2.6)

$$\alpha = \frac{\theta^2}{(1-\theta)^2 + \theta^2}. \tag{2.8}$$

An equilibrium exists if equations (2.7) and (2.8) admit a solution (θ^*, α^*) in $(0,1)^2$.

In the following theorem, we establish the existence and uniqueness of such an equilibrium. Let $\Omega = \{(p, \gamma, c, d_o, d_r) | c \in (0,1), \gamma \in [0,1), (d_o, d_r) \in [0,1]^2, p \in A\}$.

Theorem 2.1. *A unique equilibrium (θ^*, α^*) exists for each $(p, \gamma, c, d_o, d_r) \in \Omega$. Moreover, θ^* and α^* both (i) strictly increase with the cost of ownership c, commission γ and extra wear and tear cost d_o, and (ii) strictly decrease with rental price p and inconvenience cost d_r.*

Proof. We provide a proof of these results in a more general setting. We assume the usage distribution has a density function f, and f is continuous with $f(\xi) > 0$ for $\xi \in (0,1)$. Under these assumptions, $(\theta^*, \alpha^*) \in (0,1)^2$ is an equilibrium if it satisfies (2.7) and (2.6).

Observe that the right hand side of equation (2.6) is strictly increasing in θ, and the right hand side of equation (2.7) is decreasing in α, as

$$\frac{\partial\theta}{\partial\alpha} = \frac{(c-p-d_r)((1-\gamma)p-d_o)-c(1-p-d_r)}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} \leq 0.$$

This inequality holds because, on one hand, $(c-p-d_r)((1-\gamma)p-d_o)-c(1-p-d_r) = (1-p-d_r)((1-\gamma)p-d_o-c)-(1-c)((1-\gamma)p-d_o) \leq 0$ if $(1-\gamma)p-d_o \leq c$, and on the other hand, $(c-p-d_r)((1-\gamma)p-w_o)-c(1-p-d_r) = (c-p-d_r)((1-\gamma)p-d_o-c)-c(1-c) < 0$ if $(1-\gamma)p-d_o > c$. It is easy to show that (by the Intermediate Value Theorem) there exists a unique solution $(\theta^*, \alpha^*) \in (0,1)^2$ to (2.7) and (2.6) for each $(p,\gamma,c,d_o,d_r) \in \Omega$. Therefore, we denote this equilibrium by $(\theta^*(p,\gamma,c,d_o,d_r), \alpha^*(p,\gamma,c,d_o,d_r))$.

For the rest of the proof, let

$$\begin{aligned}
&h(\theta,\alpha,p,\gamma,c,d_o,d_r)\\
=&(h_1(\cdot),h_2(\cdot))\\
=&(\theta - \frac{c-((1-\gamma)p-d_o)\alpha}{p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha}, \alpha - \frac{D(\theta)}{S(\theta)+D(\theta)}),
\end{aligned}$$

and

$$g(p,\gamma,c,d_o,d_r) = (g_1(\cdot),g_2(\cdot)) = (\theta^*(\cdot),\alpha^*(\cdot)).$$

Observe that h is continuous on $[0,1]^2 \times \Omega$ unless $(\alpha,p,\gamma,d_o,d_r) = (1,1,0,0,0)$ or $(\alpha,p,d_r) = (0,0,0)$, and that $g(\cdot)$ is the unique solution to $h(g(\cdot),p,\gamma,c,d_o,d_r) = 0$ in $(0,1)^2$. It is easy to show that (by the subsequence principle) $g(\cdot)$ is continuous on Ω.

To show g is continuously differentiable on Ω° (the interior of Ω), we use Euler's notation D for differential operators. For any component x and y of the function

h, we have

$$D_{(x,y)}h = \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{pmatrix}.$$

It follows that

$$D_{(\theta,\alpha)}h = \begin{pmatrix} 1 & \frac{c(1-p-d_r)-(c-p-d_r)((1-\gamma)p-d_o)}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} \\ -\frac{\theta f(\theta)S(\theta)+(1-\theta)f(\theta)D(\theta)}{(S(\theta)+D(\theta))^2} & 1 \end{pmatrix}$$

As shown earlier, $\frac{c(1-p-d_r)-(c-p-d_r)((1-\gamma)p-d_o)}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} > 0$ on Ω°. Therefore, $D_{(\theta,\alpha)}h$ is always invertible. By the Implicit Function Theorem, g is continuously differentiable, and for each component x,

$$D_x g = -[D_{(\theta,\alpha)}h]^{-1} D_x h, \tag{2.9}$$

where

$$[D_{(\theta,\alpha)}h]^{-1} = \frac{1}{\det(D_{(\theta,\alpha)}h)} \begin{pmatrix} 1 & -\frac{c(1-p-d_r)-(c-p-d_r)((1-\gamma)p-d_o)}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} \\ \frac{\theta f(\theta)S(\theta)+(1-\theta)f(\theta)D(\theta)}{(S(\theta)+D(\theta))^2} & 1 \end{pmatrix}.$$

Calculating $D_x h$ for each component x leads to

$$D_\gamma h = \begin{pmatrix} \frac{p\alpha(c-p-d_r-(1-p-d_r)\alpha)}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} \\ 0 \end{pmatrix}, D_c h = \begin{pmatrix} -\frac{1}{p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha} \\ 0 \end{pmatrix},$$

$$D_{d_o} h = \begin{pmatrix} \frac{\alpha(c-p-d_r-(1-p-d_r)\alpha)}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} \\ 0 \end{pmatrix}, D_{d_r} h = \begin{pmatrix} \frac{(1-\alpha)(c-((1-\gamma)p-d_o)\alpha)}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} \\ 0 \end{pmatrix},$$

and

$$D_p h = \begin{pmatrix} \frac{(1-\gamma)\alpha(\alpha-c)+c(1-\alpha)+(1-\gamma)\alpha(1-\alpha)d_r+\alpha(1-\alpha)d_o}{(p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha)^2} \\ 0 \end{pmatrix}.$$

It is easy to see that $(1-\gamma)\alpha(\alpha-c)+c(1-\alpha) = ((1-\gamma)\alpha-c)(\alpha-c)+c(1-c)$ is strictly postitive for both $\alpha > c$ and $\alpha \leq c$. It is also clear from (2.7) that, in equilibrium, we have $c - ((1-\gamma)p - d_o)\alpha > 0$ and $c - p - d_r - (1-p-d_r)\alpha < 0$. Therefore, $D_p h > 0$, $D_\gamma h < 0$, $D_c h < 0$, $D_{d_o} h < 0$ and $D_{d_r} h > 0$ in equilibrium. So, we conclude $\frac{\partial \theta^*}{\partial p} < 0$, $\frac{\partial \alpha^*}{\partial p} < 0$, $\frac{\partial \theta^*}{\partial \gamma} > 0$, $\frac{\partial \alpha^*}{\partial \gamma} > 0$, $\frac{\partial \theta^*}{\partial c} > 0$, $\frac{\partial \alpha^*}{\partial c} > 0$, $\frac{\partial \theta^*}{\partial d_o} > 0$, $\frac{\partial \alpha^*}{\partial d_o} > 0$, $\frac{\partial \theta^*}{\partial d_r} < 0$, and $\frac{\partial \alpha^*}{\partial d_r} < 0$. $\square$

Let ω^* and q^* denote the corresponding ownership and total usage in equilibrium. Then,

$$\omega^* = 1 - \theta^*$$

and

$$q^* = \frac{1 - \alpha^* \theta^{*2}}{2},$$

where the expression for q^* follows from noting that $q^* = \int_{[\theta^*,1]} \xi \, d\xi + \beta \int_{[0,\theta^*]} \xi \, d\xi$ (note that total usage is the sum of usage from the owners and the fraction of usage from the non-owners that is satisfied through renting).

The following proposition describes how ownership and usage in equilibrium vary with the problem's parameters.

Proposition 2.2. *In equilibrium, ownership ω^* and usage q^* both strictly increase in price p and inconvenience cost d_r, and strictly decrease in cost of ownership c, commission γ and extra wear and tear cost d_o.*

Proof. It follows from Theorem 2.1 and the fact that both $\omega^* = 1 - \theta^*$ and $q^* = \frac{1-\alpha^*\theta^{*2}}{2}$ are decreasing in θ^* and α^*. □

While the monotonicity results in Proposition 2.2 are perhaps expected, it is not clear how ownership and usage under collaborative consumption compare to those under no collaborative consumption. In the following subsection, we provide comparisons between systems with and without collaborative consumption, and address the questions of whether or not collaborative consumption reduces product ownership and usage.

2.4.1 Impact of Collaborative Consumption on Ownership and Usage

In the absence of collaborative consumption, an individual would own a product if $u(\xi) \geq c$ and would not otherwise. Let $\hat{\theta}$ denote the solution to $u(\xi) = c$. Then, the fraction of the population that corresponds to owners (ownership) is given by

$$\hat{\omega} = \int_{[\hat{\theta},1]} f(\xi)\, d\xi = 1 - c,$$

with an associated usage given by

$$\hat{q} = \int_{[\hat{\theta},1]} \xi f(\xi)\, d\xi = \frac{1-c^2}{2}.$$

In the following proposition, we compare ownership level with and without collaborative consumption. Without loss of generality, we assume here (and in the rest of the paper) that $\frac{d_o}{1-\gamma} < 1 - d_r$ so that the set of admissible prices consists of more than a single price.

Proposition 2.3. *There exists* $p_\omega \in (\frac{d_o}{1-\gamma}, 1-d_r)$ *such that* $\omega^* = \hat{\omega}$ *if* $p = p_\omega$, $\omega^* < \hat{\omega}$ *if* $p < p_\omega$, *and* $\omega^* > \hat{\omega}$ *otherwise. Moreover,* $\frac{\partial p_\omega}{\partial \gamma} > 0$, $\frac{\partial p_\omega}{\partial c} < 0$, $\frac{\partial p_\omega}{\partial d_o} > 0$, *and* $\frac{\partial p_\omega}{\partial d_r} < 0$.

Proof. By Equations (2.7) and (2.8), θ^* is the unique solution to

$$\phi(\theta, p) = [(1+p+d_r) - ((1-\gamma)p - d_o)]\theta^3 - [2(p+d_r+c) - ((1-\gamma)p - d_o)]\theta^2 + (p+d_r+2c)\theta - c = 0 \tag{2.10}$$

in $(0,1)$. Replacing θ by c in (2.10) leads to $\phi(c,p) = c(1-c)[(1-\gamma c)p - ((1-d_r)(1-c) + d_o c)] = 0$. This implies $p_\omega = \frac{(1-d_r)(1-c)+d_o c}{1-\gamma c}$ is the rental price that induces $\theta^* = c$, or equivalently, $\omega^* = \hat{\omega}$. As we assume $\frac{d_o}{1-\gamma} < 1-d_r$, it is easy to verify that $p_\omega \in (\frac{d_o}{1-\gamma}, 1-d_r)$. This implies that it is always possible to induce $\omega^* = \hat{\omega}$. As ω^* is strictly increasing in p, the first statement follows. In addition, we have $\frac{\partial p_\omega}{\partial \gamma} = \frac{c((1-d_r)(1-c)+d_o c)}{(1-\gamma c)^2} > 0$, $\frac{\partial p_\omega}{\partial c} = \frac{d_o - (1-\gamma)(1-d_r)}{(1-\gamma c)^2} < 0$, $\frac{\partial p_\omega}{\partial d_o} = \frac{c}{1-\gamma c} > 0$, and $\frac{\partial p_\omega}{\partial d_r} = \frac{c-1}{1-\gamma c} < 0$. □

Proposition 2.3 shows that depending on the rental price p, collaborative consumption can result in either lower or higher ownership. In particular, when the rental price p is sufficiently high (above the threshold p_ω), collaborative consumption leads to higher ownership (e.g., more cars). Moreover, the threshold p_ω is decreasing in the cost of ownership c and renter's inconvenience d_r, and increasing in the commission rate γ and extra wear and tear cost d_o. The fact that p_ω is decreasing in c is perhaps surprising as it shows that collaborative consumption is more likely to lead to more ownership (and not less) when the cost of owning is high. This can be explained as follows. In the absence of collaborative consumption, when the cost of ownership is high, there are mostly non-owners. With the introduction of collaborative consumption, owning becomes more affordable as rental income subsidizes the high cost of ownership. In that case, even at low rental prices, there are individuals (those with high usage) who would switch to being owners. This switch is made more attractive by the high probability of finding a renter (given the high fraction of renters in the population). On the other hand,when the cost of ownership is low, only individuals with low usage are non-owners. For collaborative consumption to turn these non-owners into owners and lead to higher ownership, the rental price needs to be high. This is also needed to compensate for the low probability of finding a renter.

Similarly, usage can be either lower or higher with collaborative consumption than without it. In this case, there is again a price threshold p_q above which usage is higher with collaborative consumption, and below which usage is higher without collaborative consumption. When either d_o or d_r is sufficiently high, collaborative consumption always leads to higher usage. The result is stated in Proposition 2.4.

Proposition 2.4. *There exists $t \in (0,1)$ such that (i) if $\frac{d_o}{1-\gamma} + d_r < t$, then there exists $p_q \in (\frac{d_o}{1-\gamma}, 1-d_r)$ such that $q^* = \hat{q}$ if $p = p_q$, $q^* < \hat{q}$ if $p < p_q$, and $q^* > \hat{q}$ if $p > p_q$; (ii) otherwise, $q^* \geq \hat{q}$ for all $p \in [\frac{d_o}{1-\gamma}, 1-d_r]$.*

Proof. From Proposition 2.3, we have $q^*(p_\omega) = \frac{1-\alpha^* c^2}{2} > \frac{1-c^2}{2} = \hat{q}$. As q^* is continuously increasing in p, we know that either (i) or (ii) is true. Moreover, if we let $\underline{p} = \frac{d_o}{1-\gamma}$ be the minimal admissible price. Then, (i) is true if and only if $q^*(\underline{p}) < \hat{q}$; Otherwise, (ii) is true. In the rest of the proof, we show that there exists some $t \in (0,1)$ such that (i) $q^*(\underline{p}) < \hat{q}$ if $\frac{d_o}{1-\gamma} + d_r < t$, and (ii) $q^*(\underline{p}) \geq \hat{q}$ if $\frac{d_o}{1-\gamma} + d_r \geq t$.

To this end, we first show that $q^*(\underline{p})$ is strictly increasing in $\frac{d_o}{1-\gamma} + d_r$. As q^* strictly decreases in θ^*, it suffices to show that $\theta^*(\underline{p})$ is strictly decreasing in $\frac{d_o}{1-\gamma} + d_r$. From (2.10), we have

$$\phi(\theta, \underline{p}) = \theta^3 - 2c\theta^2 + 2c\theta - c + (\frac{d_o}{1-\gamma} + d_r)(\theta^3 - 2\theta^2 + \theta).$$

It is clear that $\phi(\theta, \underline{p})$ is strictly increasing in $\frac{d_o}{1-\gamma} + d_r$ for $\theta \in (0, 1)$. Therefore, $\theta^*(\underline{p})$, the unique solution to $\phi(\theta, \underline{p}) = 0$, must be strictly decreasing in $\frac{d_o}{1-\gamma} + d_r$.

We next show that $q^*(\underline{p}) > \hat{q}$ if $\frac{d_o}{1-\gamma} + d_r = 1$ (this corresponds to when $\frac{d_o}{1-\gamma} + d_r$ is at its largest), and $q^*(\underline{p}) < \hat{q}$ if $\frac{d_o}{1-\gamma} + d_r = 0$ (this corresponds to when $\frac{d_o}{1-\gamma} + d_r$ is at its smallest). Substituting α^* by θ^* yields $q^*(p) = \frac{1}{2} - \frac{\theta^{*4}}{2(2\theta^{*2} - 2\theta^* + 1)}$. It follows that $q^*(p) < \hat{q}$ if and only if $\psi(p) = \theta^{*4} - 2c^2\theta^{*2} + 2c^2\theta^* - c^2 > 0$. Note that $\phi(\theta^*(\underline{p}), \underline{p}) = 0$. Therefore, $\psi(\underline{p}) = \psi(\underline{p}) - c\phi(\theta^*(\underline{p}), \underline{p}) = \theta^{*4} - c\theta^{*3} - c(\frac{d_o}{1-\gamma} + d_r)(\theta^{*3} - 2\theta^{*2} + \theta^*)$. Then, if $\frac{d_o}{1-\gamma} + d_r = 1$, we have $\psi(\underline{p}) = \theta^*(\theta^{*3} - 2c\theta^{*2} + 2c\theta^* - c) < \theta^*\phi(\theta^*(\underline{p}), \underline{p}) = 0$. On the other hand, if $\frac{d_o}{1-\gamma} + d_r = 0$, we have $\psi(\underline{p}) = \theta^{*4} - c\theta^{*3} > 0$, since, by Proposition 2.3, $\theta^*(\underline{p}) > c$.

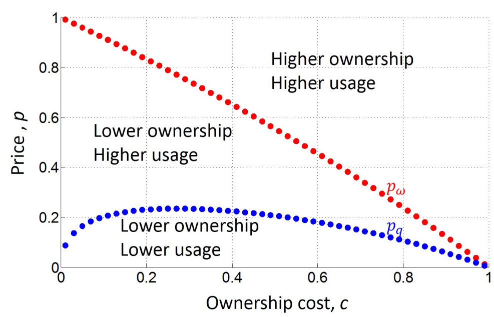
Finally, as $q^*(\underline{p})$ is strictly increasing in $\frac{d_o}{1-\gamma} + d_r$, by the Intermediate Value Theorem, there exists $t \in (0, 1)$ such that $q^*(\underline{p}) < \hat{q}$ if $\frac{d_o}{1-\gamma} + d_r < t$ and $q^*(\underline{p}) \geq \hat{q}$ if $\frac{d_o}{1-\gamma} + d_r \geq t$. This completes the proof. $\square$

Unlike p_ω, the price threshold p_q is not monotonic in c (see Figure 2.1). As c increases, p_q first increases then decreases. To understand the reason, note that collaborative consumption can lead to higher usage due to the new usage from non-owners. On the other hand, it can lead to lower usage if ownership decreases sufficiently (certainly to a level lower than that without collaborative consumption) such that the decrease in usage from those who switch from owning to renting is larger than the increase in usage from those who are non-owners. This implies that lower usage is less likely to happen if either (*i*) the individuals who switch from owning to renting can fulfill most of their usage via renting, or (*ii*) the usage from non-owners is high. The first scenario is true when the population of owners is high (i.e., the cost of ownership is low), whereas the second scenario is true when the population of non-owners is high (i.e., cost of ownership is high). Therefore, collaborative consumption is less likely to lead to lower usage when the cost of ownership is either very low or very high. Hence, the threshold p_q is first increasing then decreasing in c. When the cost of ownership is moderate, there is a balance of owners and non-owners without collaborative consumption, allowing for ownership to sufficiently decrease with relatively moderate rental prices, which in turn leads to lower usage and, correspondingly, a relatively higher threshold p_q.

The following corollary to Propositions 3 and 4 summarizes the joint impact of p and c on ownership and usage.

Corollary 2.5. *In settings where p_ω and p_q are well defined (per Propositions 3 and 4), collaborative consumption leads to higher ownership and higher usage when $p > p_\omega$, lower ownership but higher usage when $p_q < p \leq p_\omega$, and lower ownership and lower usage when $p \leq p_q$.*

Corollary 2.5, along with Propositions 3 and 4, show how price thresholds p_ω and p_q segment the full range of values of c and p into three regions, in which collaborative consumption leads to (*i*) lower ownership and lower usage, (*ii*) lower ownership but higher usage, and (*iii*) higher ownership and higher usage. Thes three regions are illustrated in Figure 2.1. These results highlight the fact that the impact of collaborative consumption on ownership and usage is perhaps more nuanced than what is sometimes claimed by advocates of collaborative consumption. The results could have implications for public policy. For example, in regions where the cost of ownership is high, the results imply that, unless rental prices are kept sufficiently low or the commission extracted by the platform is made sufficiently high, collaborative consumption would lead to more ownership and more usage. This could be an undesirable outcome if there are negative externalities associated with ownership and usage. Higher usage also implies less usage of the outside option (e.g., less use of public transport).



(a) $\gamma = 0.2$, $d_o = 0$, $d_r = 0$

Figure 2.1: Ownership and usage for varying rental prices and ownership costs

2.4.2 Impact of Collaborative Consumption on Consumers

Next, we examine the impact of collaborative consumption on consumer payoff. Consumer payoff is of course always higher with the introduction of collaborative consumption (consumers retain the option of either owning or not owning, but now enjoy the additional benefit of earning rental income if they decide to own, or of fulfilling some of their usage through renting if they decide not to own). What is less clear is who, among consumers with different usage levels, benefit more from collaborative consumption.

Proposition 2.6. *Let $\pi^*(\xi)$ and $\hat{\pi}(\xi)$ denote respectively the consumer payoff with and without collaborative consumption. Then, the difference in consumer payoff $\pi^*(\xi) - \hat{\pi}(\xi)$ is positive, piecewise linear, strictly increasing on $[0, c)$, and strictly decreasing on $[c, 1]$.*

Proof. We have

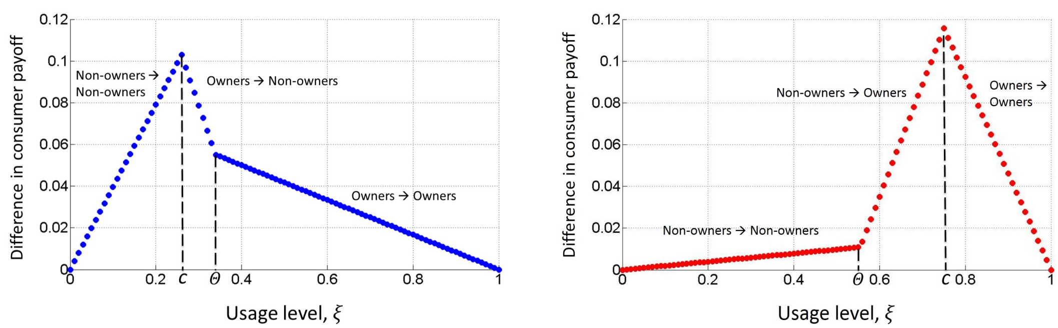
$$\pi^*(\xi) - \hat{\pi}(\xi) = \begin{cases} (1-\alpha^*)\xi(1-p-d_r) & \text{for} \quad 0 \le \xi < c; \\ -\alpha^*\xi - (1-\alpha^*)\xi(p+d_r) + c & \text{for} \quad c \le \xi < \theta^*; \\ (1-\xi)\alpha^*[(1-\gamma)p - d_o] & \text{for} \quad \theta^* \le \xi \le 1, \end{cases}$$

if $\theta^* \ge c$, and

$$\pi^*(\xi) - \hat{\pi}(\xi) = \begin{cases} (1-\alpha^*)\xi(1-p-d_r) & \text{for} \quad 0 \le \xi < \theta^*; \\ \xi + (1-\xi)\alpha^*[(1-\gamma)p - d_o] - c & \text{for} \quad \theta^* \le \xi < c; \\ (1-\xi)\alpha^*[(1-\gamma)p - d_o] & \text{for} \quad c \le \xi \le 1, \end{cases}$$

if $\theta^* < c$. As individuals retain the option of not paticipating, it is clear that $\pi^*(\xi) - \hat{\pi}(\xi)$ is positive. It is also easy to see that $\pi^*(\xi) - \hat{\pi}(\xi)$ is piecewise linear, increasing on $[0, c)$, and decreasing on $[c, 1]$. □

An important implication from Proposition 2.6 (from the fact that the difference in consumer surplus $\pi^*(\xi) - \hat{\pi}(\xi)$ is strictly increasing on $[0, c)$ and strictly decreasing on $[c, 1]$) is that consumers who benefit the most from collaborative consumption are those who are indifferent between owning and not owning without collaborative consumption (recall that $[c, 1]$ corresponds to the population of owners in the absence of collaborative consumption). This can be explained by noting that there are always three segments of consumers (see Figure 2.2). In the case where $\theta^* \geq c$ (Figure 2.2 (a)), which corresponds to the case where ownership decreases with collaborative consumption, the first segment corresponds to consumers who are non-owners in the absence of collaborative consumption and continue to be non-owners with collaborative consumption (indicated by "non-owners→non-owners" in Figure 2.2). The benefit these consumers derive from collaborative consumption is due to fulfilling part of their usage through accessing a rented product. This benefit is increasing in their usage.



(a) $\theta^* > c$, $p = 0.5$, $\gamma = 0.2$, $c = 0.27$, $d_o = d_r = 0$

(b) $\theta^* < c$, $p = 0.95$, $\gamma = 0.2$, $c = 0.75$, $d_o = d_r = 0$

Figure 2.2: Impact of usage level on the difference in consumer payoff

The second segment corresponds to consumers who are owners in the absence of collaborative consumption and switch to being non-owners with collaborative consumption (indicated by "owners→non-owners"). These consumers have to give up the fulfillment of some usage (because a rental product may not always be available) and the amount they give up is increasing in their usage. Therefore, the amount of benefit they receive from renting decreases in their usage level. The third segment consists of consumers who are owners in the absence of collaborative consumption and continue to be owners with collaborative consumption (indicated by "owners→owners"). The benefit they experience is due to rental income. This income is decreasing in their usage (they have less capacity to rent when they have more usage). A similar explanation can be provided for the case where $\theta^* < c$ (Figure 2.2 (b)).

2.5 The Platform's Problem

In this section, we consider the problem faced by the platform. We first consider the case of a for-profit platform whose objective is to maximize the revenue from successful transactions. Then, we consider the case of a not-for-profit platform (e.g., a platform owned by a non-profit organization, government agency, or municipality) whose objective is to maximize social welfare.[6] We compare the outcomes of these platforms in terms of ownership, usage and social welfare. We also benchmark the social welfare of these platforms against the maximum feasible social welfare.

A platform may decide, among others, on the price and commission rate. In this section, we focus on price as the primary decision made by the platform and

[6]An example of a not-for-profit platform is NeighborGoods, a peer-to-peer platform that facilitates the sharing of household goods. NeighborGoods allows owners to earn a rental fee but does not extract for itself a commission rate.

treat other parameters as being exogenously specified (a survey of major peer-to-peer car sharing platforms worldwide reveals that commission rates fall mostly within a relatively narrow range, from 30% to 40% for those that include insurance, and do not typically vary across markets in which platforms operate). There are of course settings where the price is a decision made by the owners. Price may then be determined through a market clearing mechanism (i.e., the price under which supply equals demand; see for example Jiang and Tian [2016]). In our case, because of friction in matching supply and demand, the supply-demand balance equation in (2.4) can, per Theorem 1, be satisfied by any feasible price. Thus, the market clearing price is not unique and the system may settle on a price that maximizes neither social welfare nor platform revenue. Moreover, as we show in Section 5.1, platform revenue (or social welfare) can be highly sensitive to price, giving the platform an incentive to optimize price. Platform pricing may also be beneficial to owners as it can serve as a coordinating tool and reduce competition among them. More significantly, and as we show in Section 2.5.2, the social welfare that results from a for-profit platform tends to be close to that resulting from a not-for-profit platform.

In what follows, we provide detailed analysis for the for-profit and not-for-profit platforms under the assumptions of Section 2.4. In Sections 2.5.1 to 2.5.3, we consider the case where $(d_o, d_r) = (0, 0)$. In Section 2.5.4, we discuss the case where $(d_o, d_r) \neq (0, 0)$.

2.5.1 The For-Profit Platform

For a for-profit platform, the objective is to maximize $\gamma p \alpha S(\theta)$ the commission income generated from the fraction of supply that is matched with demand. In particular, the platforms optimization problem can be stated as follows.

$$\max_{p} \quad v_r(p) = \gamma p \alpha S(\theta) \tag{2.11}$$

$$\text{subject to} \quad \pi_o(\theta) = \pi_r(\theta), \tag{2.12}$$

$$\alpha = \frac{D(\theta)}{D(\theta) + S(\theta)}, \tag{2.13}$$

$$p \geq \frac{d_o}{1-\gamma}, \text{ and} \tag{2.14}$$

$$p \leq 1 - d_r. \tag{2.15}$$

The constraints (2.12) and (2.13) are the defining equations for the equilibrium (θ^*, α^*). Constraints (2.14) and (2.15) ensure that price is in the feasible set A. In what follows, we assume that $\gamma > 0$ (the platform's revenue is otherwise always zero).

Under the assumptions of Section 2.4, the for-profit platform's problem can be restated as follows:

$$\max_p \quad v_r(p) = \tfrac{1}{2}\gamma p \alpha (1-\theta)^2 \tag{2.16}$$

subject to (2.7) and (2.8) and $p \in A$. It is difficult to analyze (2.16) directly. However, as the map between θ and p is bijective, we can use (2.7) and (2.8) to express p in terms of θ as

$$p(\theta) = \frac{-\theta^3 + 2c\theta^2 - 2c\theta + c}{\theta(\theta - 1)(\gamma\theta - 1)}. \tag{2.17}$$

Hence, (2.16) can be expressed as

$$\max_\theta \quad v_r(\theta) = \tfrac{\gamma}{2}\tfrac{(1-\theta)\theta(\theta^3-2c\theta^2+2c\theta-c)}{((1-\theta)^2+\theta^2)(\gamma\theta-1)} \quad \text{subject to } \theta \in [\underline{\theta}, \overline{\theta}] \tag{2.18}$$

where $\underline{\theta}$ is the solution to (2.7) and (2.8) at $p = 1$, $\overline{\theta}$ is the solution at $p = 0$, and $[\underline{\theta}, \overline{\theta}]$ is the set of solutions induced by $p \in [0, 1]$. We can use (2.17) to verify whether θ is in $[\underline{\theta}, \overline{\theta}]$. Specifically, $\theta < \underline{\theta}$ if $p(\theta) > 1$, $\theta \in [\underline{\theta}, \overline{\theta}]$ if $p(\theta) \in [0, 1]$, and $\theta > \overline{\theta}$ if $p(\theta) < 0$.

Proposition 2.7. *$v_r(\theta)$ is strictly quasiconcave in θ.*

Proposition 2.7 shows that the platform's problem is not difficult to solve. Depending on the value of γ and c, $v_r(\theta)$ is either decreasing or first increasing then decreasing on $[\underline{\theta}, \overline{\theta}]$. In both cases, the optimal solution to (2.18), which we denote by θ_r^*, is unique; see Appendix 2.B for a proof of this and all subsequent results. We let p_r^*, ω_r^*, and q_r^* denote the corresponding price, ownership, and usage, respectively. We also use the notation v_r^* to denote the optimal revenue $v_r(\theta_r^*)$.

Proposition 2.8. *The platform's optimal revenue, v_r^*, is strictly quasiconcave in c, first strictly increasing and then strictly decreasing.*

Proposition 2.8 suggests that a platform would be most profitable when the cost of ownership is "moderate" and away from the extremes of being either very high or very low. In these extreme cases, not enough transactions take place because of either not enough renters (when the cost of ownership is low) or not enough owners (when the cost of ownership is high). This is perhaps consistent with the experience of iCarsclub, a peer-to-peer car sharing platform, that was first launched in Singapore, a country where the cost of ownership is exceptionally high and car ownership is low. iCarsclub struggled in Singapore and had to temporarily suspend operations. However, it is thriving in China where it operates under the name PPzuche and is present in several cities (Clifford Teo, CEO of iCarsclub, personal communication, 2015). This result also implies that a platform may have an incentive to affect the cost of ownership. For example, when the cost of ownership is low, a platform may find it beneficial to impose a fixed membership fee on owners, increasing the effective cost of ownership. On the other hand, when the cost of ownership is high, the platform may find it beneficial to lower the effective cost of ownership by offering, for example, subsidies (or assistance with financing) toward the purchase of new products.

Proposition 2.9. *There exists a threshold $c_{r,\omega} \in (0,1)$ such that optimal ownership $\omega_r^* = \hat{\omega}$ if $c = c_{r,\omega}$, $\omega_r^* < \hat{\omega}$ if $c < c_{r,\omega}$, and $\omega_r^* > \hat{\omega}$ otherwise, with $c_{r,\omega}$ strictly increasing in γ.*

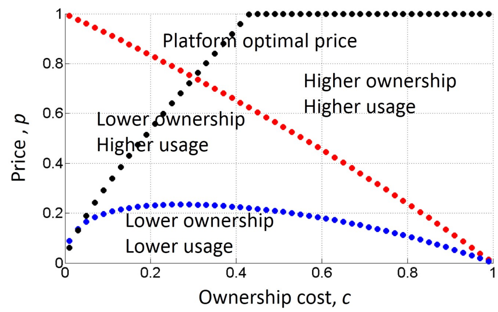
Proposition 2.9 shows that it continues to be possible, even when the price is chosen optimally by a revenue maximizing platform, for collaborative consumption to lead to either higher or lower ownership. In particular, collaborative consumption leads to higher ownership when the cost of ownership is sufficiently high (above the threshold $c_{r,\omega}$) and to lower ownership when the cost of ownership is sufficiently low (below the threshold $c_{r,\omega}$). This can be explained as follows. The platform has an incentive to somewhat balance supply and demand (otherwise few rentals will take place). When the cost of ownership is high, ownership is low in the absence of collaborative consumption. In this case, the platform would try to induce, via higher prices, higher ownership, so as to generate more supply (hence, the result that a sufficiently high cost of ownership leads to higher ownership under collaborative consumption).[7] Similarly, when the cost of ownership is low, the platform would try to induce lower ownership via lower prices, so as to generate more demand (hence, the result that a sufficiently low cost of ownership leads to low ownership under collaborative consumption).

We also observe that usage under platform pricing can be either higher or lower than that without collaborative consumption. Again, there exists a threshold $c_{r,q} < c_{r,\omega}$ in the cost of ownership, below which collaborative consumption leads to lower usage and above which collaborative consumption leads to higher usage. The impact of ownership cost on product ownership and usage under platform pricing is illustrated in Figure 2.3 where the dashed black line corresponds to the optimal price.

2.5.2 The Not-for-Profit Platform

For a not-for-profit platform, the objective is to maximize social welfare (i.e., the sum of consumer surplus and platform revenue). Thus, the platform's problem

[7]This perhaps validates concerns expressed by the Singapore authorities that allowing peer-to-peer car sharing would increase car usage and road congestion and their initial decision to restrict peer-to-peer car rentals to evenings and weekends (Clifford Teo, CEO of iCarsclub, personal communication, 2015).



(a) $\gamma = 0.2$, $d_o = 0$, $d_r = 0$

Figure 2.3: Impact of ownership cost on ownership and usage

can be stated as

$$\max_p \quad v_s(p) = \int_{[\theta,1]} (\xi - c) f(\xi)\, d\xi + \int_{[0,\theta)} \beta \xi f(\xi)\, d\xi, \tag{2.19}$$

subject to constraints (2.12)-(2.15).

Under the assumptions of Section 2.4, the platform's problem can be restated as follows:

$$\max_p \quad v_s(p) = \tfrac{1}{2}(1 - \alpha\theta^2) - (1-\theta)c \tag{2.20}$$

subject to (2.7) and (2.8) and $p \in A$, or equivalently as

$$\max_\theta \quad v_s(\theta) = \tfrac{1}{2}\left(1 - \tfrac{\theta^4}{(1-\theta)^2+\theta^2}\right) - (1-\theta)c \quad \text{subject to } \theta \in [\underline{\theta}, \overline{\theta}]. \tag{2.21}$$

Analysis and results similar to those obtained for the for-profit platform can be obtained for the not-for-profit platform. In particular, we can show that the social welfare function, v_s, is strictly concave in θ, indicating that computing the optimal solution for the not-for-profit platform is also not difficult (we omit the details for the sake of brevity). The result also implies that (2.21) admits a unique optimal solution, which we denote by θ_s^*, with a resulting optimal social welfare which we denote by v_s^*.

The following proposition characterizes θ_s^* for varying values of γ.

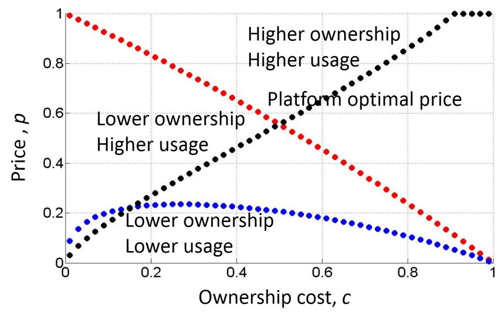
Proposition 2.10. *There exists a strictly positive decreasing function $\gamma_s(c)$ such that $\theta_s^* \in (\underline{\theta}, \overline{\theta})$ if $\gamma < \gamma_s$, and $\theta_s^* = \underline{\theta}$ otherwise. Consequently, if $\gamma \leq \gamma_s(c)$, then*

$$\max_{\theta \in [\underline{\theta}, \overline{\theta}]} v_s = \max_{\theta \in [0,1]} v_s.$$

Proposition 2.10 shows that θ_s^* is an interior solution (satisfying $\frac{\partial v_s}{\partial \theta}(\theta_s^*) = 0$) if the commission rate is sufficiently low (below the threshold γ_s). Otherwise, it is the boundary solution $\underline{\theta}$. In particular, θ_s^* could never take the value of $\overline{\theta}$. An important implication of this result is that, when $\gamma < \gamma_s$, a not-for-profit platform that relies on price alone as a decision variable would be able to achieve the maximum feasible social welfare (i.e., the social welfare that would be realized by a social planner who can directly decide on the fraction of non-owners, θ). Note that this is especially true if the not-for-profit platform does not charge a commission rate (i.e., $\gamma = 0$).

Similar to the case of the for-profit platform, we can also show that a not-for-profit platform can lead to either higher or lower ownership or usage (relative to the case without collaborative consumption). Again, there are thresholds $c_{s,q} < c_{s,\omega}$ in the cost of ownership such that (i) ownership and usage are both lower if $c \leq c_{s,q}$, (ii) ownership is lower but usage is higher if $c_{s,q} < c \leq c_{s,\omega}$, and (iii) ownership and usage are both higher if $c \geq c_{s,\omega}$. The impact of ownership cost on product ownership and usage under platform pricing is illustrated in Figure 2.4.

In the following proposition, we compare outcomes under the for-profit and



(a) $\gamma = 0.2$, $d_o = 0$, $d_r = 0$

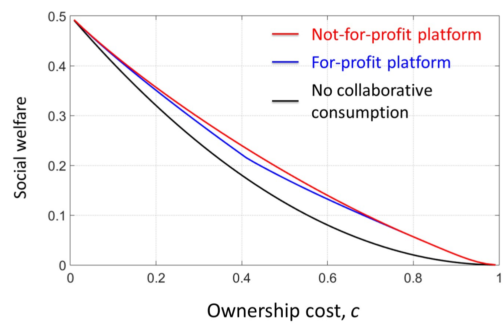
Figure 2.4: Impact of ownership cost on ownership and usage

not-for-profit platforms. In particular, we show that a not-for-profit platform would always charge a lower price than a for-profit platform. Therefore, it would also induce lower ownership and lower usage.

Proposition 2.11. *Let p_s^*, ω_s^* and q_s^* denote the optimal price, ownership and usage levels under a not-for-profit platform, respectively. Then, $p_s^* \leq p_r^*$, $\omega_s^* \leq \omega_r^*$, and $q_s^* \leq q_r^*$.*

A not-for-profit platform induces lower ownership by charging lower prices because it accounts for the negative impact of the cost of ownership on social welfare. In settings where there are negative externalities associated with ownership and usage, the result in Proposition 2.11 shows that, even without explicitly accounting for these costs, the not-for-profit platform would also lower such externalities (since both ownership and usage are lower). The fact that social welfare is maximized at prices lower than those that would be charged by a for-profit platform suggests that a regulator may be able to nudge a for-profit platform toward outcomes with higher social welfare by putting a cap on price.

Figure 2.5 illustrates the differences in social welfare between a system without collaborative consumption and systems with collaborative consumption under (a) a for-profit platform (a revenue-maximizing platform) and (b) a not-for-profit platform (a social welfare-maximizing platform). Systems with collaborative consumption can improve social welfare substantially, especially when the cost of ownership is neither too high nor too low (in those extreme cases, there are either mostly owners or mostly renters and, therefore, few transactions). However, the differences in social welfare between the for-profit and not-for-profit platforms are not very significant. This is because both platforms have a similar interest in maintaining a relative balance of renters and owners.



(a) $\gamma = 0.2$, $d_o = 0$, $d_r = 0$

Figure 2.5: Impact of ownership cost on social welfare

2.5.3 Systems with Negative Externalities

In this section, we consider settings where there are negative externalities associated with either usage or ownership. In that case, social welfare must account for the additional cost of these externalities. In particular, the following additional terms must be subtracted from the expression of social welfare in (2.19)

$$e_q q(\theta) + e_\omega \omega(\theta), \tag{2.22}$$

or equivalently

$$e_q\Big(\int_{[\theta,1]} \xi f(\xi)\, d\xi + \beta \int_{[0,\theta)} \xi f(\xi)\, d\xi\Big) + e_\omega \int_{[\theta,1]} f(\xi)\, d\xi, \tag{2.23}$$

where e_q and e_ω correspond to the social (or environmental) cost per unit of usage and per unit of ownership, respectively. This is consistent with the so-called *lifecycle* approach to assessing the social impact of using and owning products (see for example Reap et al. [2008]). The parameter e_q accounts for the social (or environmental) cost of using a product not captured by the utility (e.g., the cost of pollution associated with using a product), while e_ω would account for the social cost of product manufacturing, distribution, and end-of-life disposal.

For a not-for-profit platform, the optimization problem can then be restated as

$$\max_\theta \quad v_e(\theta) = \tfrac{1}{2}(1-e_q)(1-\tfrac{\theta^4}{(1-\theta)^2+\theta^2}) - (c+e_\omega)(1-\theta) \quad \text{subject to } \theta \in [\underline{\theta}, \overline{\theta}]. \tag{2.24}$$

It is easy to show that the modified social welfare function v_e is still strictly quasiconcave in θ. Moreover, the optimal solution, which we denote by θ_e^*, is strictly increasing in both e_q and e_ω. As a result, the ownership and usage levels obtained under $e_q > 0$ and $e_\omega > 0$ are lower than those obtained under $e_q = e_\omega = 0$. Therefore, Proposition 2.11 continues to hold. However, Proposition 2.10 may no longer be valid if either e_q or e_ω is too large. That is, the platform may not be able to achieve the maximum feasible social welfare even if the commission rate is negligible. In this case, in order to achieve a higher social welfare, the platform may need to either subsidize non-owners (e.g., improve rental experience by reducing inconvenience cost) or penalize owners (e.g., make the ownership cost higher by charging extra tax on ownership), in addition to setting a low rental price.

We conclude this section by addressing the question of whether collaborative consumption reduces the total cost of negative externalities, $e_q q(\theta) + e_\omega \omega(\theta)$. Recall that collaborative consumption (under either a for-profit or a not-for-profit platform) leads to lower ownership and lower usage when the cost of ownership is sufficiently low, and it leads to higher ownership and higher usage when the cost of ownership is sufficiently high (see Figures 2.3 and 2.4). This implies that collaborative consumption could either decrease or increase negative externalities, with a decrease more likely when the cost of ownership is low. In numerical experiments, we observe that there exists a threshold on the cost of ownership, which we denote by c_e, such that collaborative consumption reduces negative externalities if and only if $c < c_e$. We also observe that c_e is decreasing in e_q and increasing in e_ω, indicating that collaborative consumption is more likely to reduce negative externalities if the social (or environmental) cost of using products is relatively low compared to that of owning. This is illustrated for an example system in Figure 2.6.

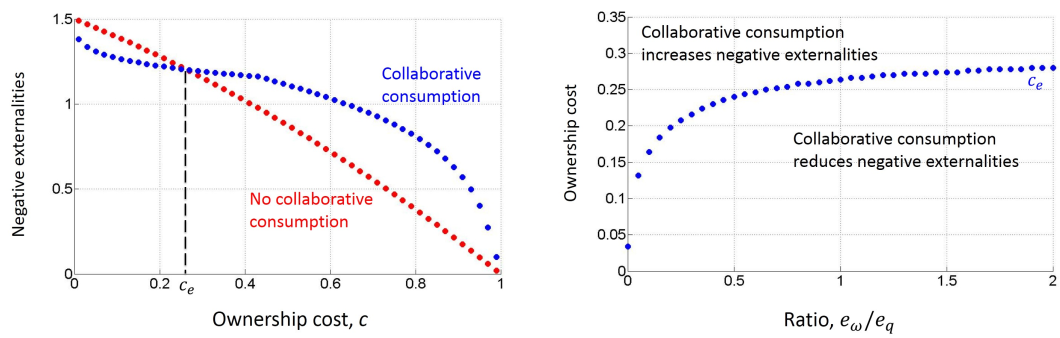


Figure 2.6: The impact of collaborative consumption on negative externalities

2.5.4 The Impact of Extra Wear and Tear and Inconvenience Costs

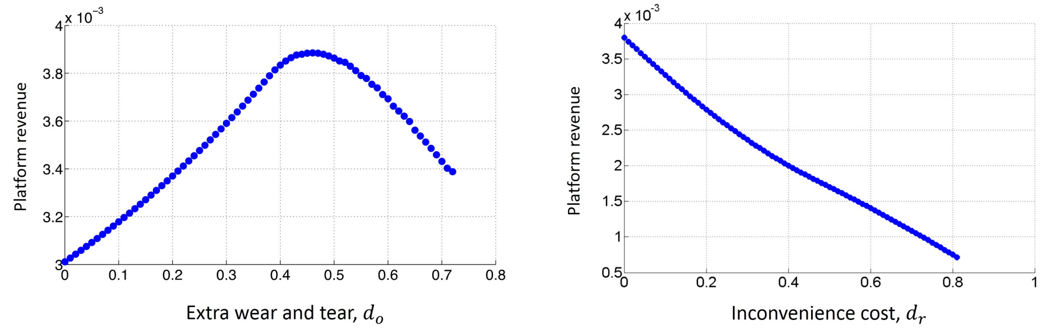
In this section, we consider the case where $(d_o, d_r) \neq 0$. The extra wear and tear cost d_o reduces the payoff of owners and, therefore, places a lower bound on the set of admissible prices: $p \geq \frac{d_o}{(1-\gamma)}$. Similarly, the inconvenience cost d_r reduces the payoff of renters and, consequently, places an upper bound on the price: $p \leq 1 - d_r$. Obtaining analytical results is difficult. However, we are able to confirm numerically that all the results obtained for $(d_o, d_r) = 0$ continue to hold (details are omitted for brevity).

Of additional interest is the impact of d_o and d_r on platform revenue and social welfare. For both the for-profit and not-for-profit platforms, we observe that social welfare is decreasing in both d_o and d_r. This is consistent with intuition. However, revenue for the for-profit platform can be non-monotonic in d_o. In particular, when the cost of ownership is low, platform revenue can first increase then decrease with d_o. This effect appears related to the fact that platform revenue is, per Proposition

8, non-monotonic in the cost of ownership. A higher value of d_o can be beneficial to the platform if it helps balance the amount of owners and renters (i.e., reduce ownership), leading to a greater amount of transactions. An implication from this result is that between two platforms, the one with the higher wear and tear cost (everything else being equal) can be more profitable. The result also implies that a for-profit platform may not always have an incentive to reduce this cost. Note that, in some cases, a platform could exert costly effort to reduce this cost. For example, when extra wear and tear is, in part, due to renters' negligence, more effort could be invested in the vetting of would-be renters. Alternatively, the platform could provide more comprehensive insurance coverage or monitor more closely the usage behavior of a renter (such monitoring technology is already available for example in the case of cars). On the other hand, the inconvenience cost d_r does not have the same effect on platform revenue. An increase in d_r could lead to more transactions. However, it limits the price a platform could charge. The net effect is that the platform revenue is always decreasing in d_r. These effects are illustrated in Figure 2.7.

2.6 Extensions

In this section, we examine the robustness of our results by extending the analysis to two important settings: (1) a setting where non-owners, in addition to renting from individual owners, have the option of renting from a third party service provider and (2) a setting where the platform, in addition to matching owners and non-owners, can also own products that it puts out for rent. In each case, we



(a) $\gamma = 0.4$, $c = 0.2$, $d_r = 0$ (b) $\gamma = 0.4$, $c = 0.8$, $d_o = 0$

Figure 2.7: Platform revenue for varying extra wear and tear and inconvenience costs

examine, how collaborative consumption affects ownership and usage, and how these outcomes vary from those observed in the original model. We also discuss other extensions and generalizations. In particular, we consider cases where (1) the extra wear and tear and inconvenience costs are heterogeneous among individuals and (2) usage, instead of being exogenously specified, is a decision individuals make, and (3) usage is exogenous but has a general distribution.

2.6.1 Systems with a Third Party Service Provider

In this section, we consider a setting where the outside option for non-owners includes renting through a third party service provider. The service provider (e.g., a car rental service) owns multiple units of the product which they make available for rent. We let $\tilde{p}$ denote the rental price and $\tilde{d}_r$ the inconvenience cost incurred by a renter when using the service (we use "~" throughout to indicate the parameters for the third party service provider). We allow for the possibility that the service is not always reliable, and we denote the fraction of time that a product is available for rent from the service provider when requested by a renter by $\tilde{\beta}$. We refer to $\tilde{\beta}$ as the *service level* of the service provider. As in the original model, we focus on the case where the utility function $u(\xi) = \xi$, and individual usage ξ follows a uniform distribution

In the case of no collaborative consumption (i.e., peer-to-peer product sharing is not available), individuals have the option of either owning the product or renting it through the service provider. An owner with usage level ξ has payoff

$$\tilde{\pi}_o(\xi) = \xi - c$$

while a renter has payoff

$$\tilde{\pi}_r(\xi) = \tilde{\beta}(1 - \tilde{p} - \tilde{d}_r)\xi.$$

Let

$$\tilde{\theta} = \begin{cases} \frac{c}{\tilde{p}+\tilde{d}_r+(1-\tilde{p}-\tilde{d}_r)(1-\tilde{\beta})} & \text{if } \tilde{\beta} \leq \frac{1-c}{1-\tilde{p}-\tilde{d}_r} \\ 1 & \text{otherwise.} \end{cases}$$

Then, individuals with usage $\xi < \tilde{\theta}$ choose to be renters, those with usage $\xi > \tilde{\theta}$ choose to be owners, and those with usage $\xi = \tilde{\theta}$ are indifferent between renting and owning. Note that when the service level is sufficiently high ($\tilde{\beta} \geq \frac{1-c}{1-\tilde{p}-\tilde{d}}$), there are no owners and everyone chooses to rent. The threshold above which the service level must be for this to occur is decreasing in the cost of ownership and increasing in the rental price and inconvenience cost.

In the presence of collaborative consumption, we assume that consumers always prefer peer-to-peer product sharing over renting through the service provider.[8]

[8]It is possible to treat the reverse case. It is also possible to consider alternative assumptions,

That is, we require $p + d_r \leq \tilde{p} + \tilde{d}_r$. A consumer would then seek to rent from the service provider when no product is available through peer-to-peer product sharing. An owner with usage ξ has payoff

$$\pi_o(\xi) = \xi + (1 - \xi)\alpha((1 - \gamma)p - d_o) - c$$

while that of a renter is given by

$$\pi_r(\xi) = \beta(1 - p - d_r)\xi - (1 - \beta)\tilde{\beta}(1 - \tilde{p} - \tilde{d}_r)\xi.$$

An equilibrium under collaborative consumption can be defined in a similar way as in Section 2.3. In particular, θ is now given by

$$\theta = \min\left(\max\left(0, \frac{c - ((1 - \gamma)p - d_o)\alpha}{(p + d_r) + [(1 - p - d_r) - \tilde{\beta}(1 - \tilde{p} - \tilde{d}_r) - ((1 - \gamma)p - d_o)]\alpha}\right), 1\right) \tag{2.25}$$

while α is again given by

$$\alpha = \frac{\theta^2}{(1 - \theta)^2 + \theta^2}. \tag{2.26}$$

An equilibrium under collaborative consumption exists if (2.25) and (2.26) admit a solution $(\theta^*, \alpha^*) \in (0, 1)^2$.

In the following theorem, we show that an equilibrium (θ^*, α^*) under collaborative consumption exists and is unique for each feasible combination of problem parameters if and only if $\tilde{\beta} < \frac{1-c}{1-\tilde{p}-\tilde{d}_r}$. Let $\Omega = \{(p, \gamma, c, d_o, d_r, \tilde{\beta}, \tilde{p}, \tilde{d}_r) | c, \tilde{\beta} \in (0, 1), \gamma \in [0, 1), d_o, d_r, \tilde{p}, \tilde{d}_r \in [0, 1], 0 \leq \tilde{p} + \tilde{d}_r \leq 1, \frac{d_o}{1-\gamma} \leq p \leq \tilde{p} + \tilde{d}_r - d_r\}$.

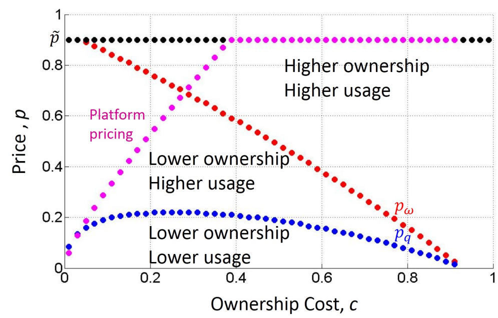
Theorem 2.12. *An equilibrium (θ^*, α^*) under collaborative consumption exists*

including the case where price and service level are endogenously determined through the dynamics of competition between the sharing platform and the service provider (we leave this as an area of future investigation).

and is unique for each $(p, \gamma, c, d_o, d_r, \tilde{\beta}, \tilde{p}, \tilde{d_r}) \in \Omega$ *if and only if* $\tilde{\beta} < \frac{1-c}{1-\tilde{p}-\tilde{d_r}}$. *In equilibrium,* $\frac{\partial \theta^*}{\partial p} < 0$, $\frac{\partial \alpha^*}{\partial p} < 0$, $\frac{\partial \theta^*}{\partial \gamma} > 0$, $\frac{\partial \alpha^*}{\partial \gamma} > 0$, $\frac{\partial \theta^*}{\partial c} > 0$, $\frac{\partial \alpha^*}{\partial c} > 0$, $\frac{\partial \theta^*}{\partial d_o} > 0$, $\frac{\partial \alpha^*}{\partial d_o} > 0$, $\frac{\partial \theta^*}{\partial d_r} < 0$, $\frac{\partial \alpha^*}{\partial d_r} < 0$, $\frac{\partial \theta^*}{\partial \tilde{\beta}} > 0$, $\frac{\partial \alpha^*}{\partial \tilde{\beta}} > 0$, $\frac{\partial \theta^*}{\partial \tilde{p}} < 0$, $\frac{\partial \alpha^*}{\partial \tilde{p}} < 0$, $\frac{\partial \theta^*}{\partial \tilde{d_r}} < 0$, *and* $\frac{\partial \alpha^*}{\partial \tilde{d_r}} < 0$. *Consequently, ownership* ω^* *and usage* q^* *are both strictly decreasing in the cost of ownership* c, *commission* γ, *and extra wear and tear cost* d_o, *but strictly increasing in rental prices* p *and* $\tilde{p}$, *and inconvenience costs* d_r *and* $\tilde{d_r}$.

Theorem 2.12 suggests that peer-to-peer product sharing can co-exist in equilibrium with a third party service provider as long as the service level offered by the service provider is sufficiently low ($\tilde{\beta} < \frac{1-c}{1-\tilde{p}-\tilde{d}}$). Otherwise, all individuals would prefer renting through the service provider to owning. As with the original model in Section 2.4, we can show that ownership ω^* and usage q^* still increase in p, d_r (also in $\tilde{p}$ and $\tilde{d_r}$), but decrease in c, γ, and d_o. More importantly, collaborative consumption can still lead to either higher or lower ownership and usage. There are again price thresholds p_ω and p_q that segment the range of values of c and p into three regions in which collaborative consumption leads to (i) lower ownership and lower usage, (ii) lower ownership but higher usage, and (iii) higher ownership and higher usage. The price threshold p_ω is again decreasing in the cost of ownership c, implying that higher ownership is more likely when the cost of ownership is higher. Hence, all the insights, regarding the impact of collaborative consumption on ownership and usage, continue to be valid. In the case where the platform chooses prices (either to maximize profit or social welfare), there are again thresholds $c_q < c_\omega$ in the cost of ownership such that (i) ownership and usage are both lower if $c \leq c_q$, (ii) ownership is lower but usage is higher if

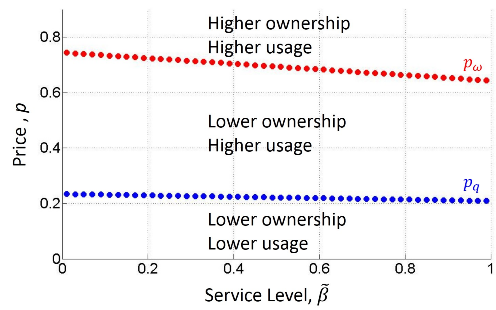
$c_q < c \leq c_\omega$, and (iii) ownership and usage are both higher if $c \geq c_\omega$. This is illustrated for an example system in Figure 2.8.



(a) $\gamma = 0.2$, $\tilde{p} = 0.9$, $\tilde{\beta} = 0.7$, $d_o, d_r, \tilde{d_r} = 0$

Figure 2.8: Ownership and usage for varying rental prices and ownership costs

In addition to confirming results consistent with the original model, we can show that the price thresholds p_ω and p_q are both decreasing in the service level $\tilde{\beta}$. This is perhaps surprising as it implies that collaborative consumption is more likely to lead to higher ownership and usage when the service level offered by the service provider is higher. This can be explained as follows. In the absence of collaborative consumption, a higher service level leads to more renters. Introducing collaborative consumption to a population with a large number of renters makes ownership attractive as the likelihood of finding a renter would be high. This is illustrated for an example system in Figure 2.9. Finally, note that the fact that p_ω and p_q are decreasing in the service level $\tilde{\beta}$ implies that these thresholds are lower with a third party service provider than without one (a system with no service provider can be viewed as one with $\tilde{\beta} = 0$).



(a) $\gamma = 0.2$, $\tilde{p} = 0.9$, $c = 0.3$, $d_o, d_r, \tilde{d_r} = 0$

Figure 2.9: The impact of price and service level on ownership and usage

2.6.2 Systems Where the Platform Owns Assets

We have so far assumed that the platform does not own products of its own and its only source of revenue is from commissions. This is consistent with the observed practice of most existing peer-to-peer product sharing platforms. Owning physical assets poses several challenges, particularly to startups, including access to capital, managerial knowhow, and the need to invest in physical infrastructure (e.g., parking lots and maintenance facilities in the case of cars). Owning, managing, and maintaining physical assets can also limit the ability of some platforms to grow rapidly. More significantly, there can be perceived business risks associated with the ownership of physical assets. Nevertheless, it is conceivable that, as these platforms grow, they can find it profitable to own products of their own. Below, we briefly describe such a setting and examine the implication of product ownership by the platform.

Consider a setting similar in all aspects to the one described in Sections 3 and 4, except that the platform may own products of its own that it puts out for rent. Let S_p and S_o denote respectively the amount of supply generated from platform- and individually-owned products. Then, the total amount of supply in the system is given by $S = S_p + S_o$. We assume that renters do not differentiate between products owned by the platform and those owned by individuals, with the same rental price applying to both (it is possible of course to consider alternative assumptions where renters may exhibit preferences for one over the other). We also assume that the platform incurs the same ownership cost (per unit of supply) as individual owners (it is easy to relax this assumption by incorporating economies/diseconomies of scale). Then, given the rental price p and level of supply S_p, the platform profit is expressed as $v(p, S_p) = \gamma p \alpha S_o + p\alpha S_p - cS_p$.

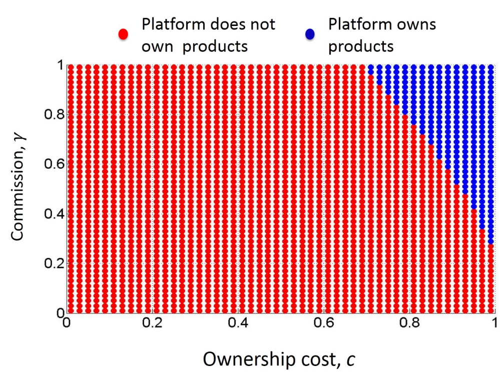
The platform now generates revenue from two sources, peer-to-peer rentals and rentals of platform-owned products (the first two terms in the above expression), while incurring the ownership cost for the products it owns (the third term in the expression). For a fixed price, the introduction of platform-owned products makes foregoing ownership more attractive, for the likelihood of successfully renting a product is now higher. Hence, individual ownership decreases and demand for rentals increases. However, it is not clear what the impact is on α (the likelihood of successfully renting out a product) since the increase in rental demand is accompanied by an injection of supply from the platform.

It turns out that, in equilibrium,[9] both the supply due to individual owners S_o and the matching parameter α are decreasing in S_p. This implies that platform

[9]We can again show that, for any value of platform supply $S_p \geq 0$, an equilibrium (θ^*, α^*) exists and is unique for every feasible combination of parameter values. We can also show that θ^* and α^* are still increasing in c, γ, d_o, but decreasing in p and d_r. In addition, θ^* increases while α^* decreases as S_p increases. The analysis is similar to those leading to Theorems 2.1 and 2.12.

revenue from peer-to-peer rentals would be maximized by the platform not owning products. Therefore, for owning products to be optimal, the platform must generate sufficient profit from the products it owns. That is, $p\alpha S_p - cS_p$ must be sufficiently large. A necessary condition for this to occur is for $p\alpha - c > 0$ when $S_p = 0$. Since α is decreasing in S_p, for relatively large values of S_p to be optimal, the price must also be relatively high.

The optimal price p^* and supply level S_p^* are difficult to characterize analytically. However, we observe numerically that the platform would own products (i.e., $S_p^* > 0$) if and only if the commission rate γ and the cost of ownership c are sufficiently large (see Figure 2.10). This is consistent with the discussion above, as $p\alpha - c > 0$ requires both γ (recall that $(1-\gamma)p\alpha - c < 0$ in the presence of collaborative consumption) and p to be large, and a large p is only optimal when c is also large.



(a) $d_o = 0$, $d_r = 0$

Figure 2.10: The platform's incentive in owning products

In summary, there seems to be a relatively narrow range of parameter values under which a platform finds it optimal to own products, since doing so cannibalizes its revenue from peer-to-peer rentals while reducing the likelihood that products, including its own, find renters. Note also that a platform that owns products must choose price p such that $p > c$. In contrast, a price $p < c$ is feasible for a pure peer-to-peer platform (because owners derive positive utility from using the product, they are willing to rent it for a price $p < c$). Hence, a platform that owns products (or relies exclusively on its own products) may invite competition from pure peer-to-peer platforms that can afford to charge a lower price (this is arguably the competitive advantage of many peer-to-peer sharing platforms).

Finally, we note that, as in the original model, the introduction of a platform (regardless of whether or not it owns products) can lead to either higher or lower ownership/usage, with higher ownership/usage again more likely when the cost of ownership is higher (this is true for both the for-profit and not-for-profit platforms). Moreover, we observe that collaborative consumption is more likely to lead to higher ownership and usage when the platform owns products.

2.6.3 Systems with Heterogeneous Extra Wear and Tear and Inconvenience Costs

In this section, we consider a setting where individuals are heterogeneous in their sensitivity to extra wear and tear and inconvenience costs. Individuals are now differentiated not only by their usage level ξ but also by their sensitivity to extra wear and tear cost d_o and inconvenience cost d_r. We let $f(\xi, d_o, d_r)$ denote the distribution of (ξ, d_o, d_r) with $f(\xi, d_o, d_r) > 0$ for $\xi, d_o, d_r \in (0, 1)$. All other assumptions remain per the original model in Sections 2.3 and 2.4. We continue to assume $p, d_o, d_r, \alpha, \beta \in [0,1]$, $\gamma \in [0,1)$ and $c \in (0,1)$.

In the presence of collaborative consumption, each individual must now decide on one of four strategies: (1) to be an owner and not participate in collaborative consumption (i.e., an owner who never puts her product up for rent), (2) to be an owner and participate in collaborative consumption, (3) to be a non-owner and participate in collaborative consumption (i.e., a non-owner who always attempts to rent a product), and (4) to be a non-owner and not participate in collaborative consumption (i.e., a non-owner who always abstains from renting). We refer to strategies (1)-(4) as ON, OP, NP and NN, respectively. The payoffs associated with each strategy for an individual with type (ξ, d_o, d_r) are given as follows:

$$\pi_{OP}(\xi, d_o, d_r) = \xi + \alpha(1-\xi)((1-\gamma)p - w) - c;$$
$$\pi_{ON}(\xi, d_o, d_r) = \xi - c;$$
$$\pi_{NP}(\xi, d_o, d_r) = \beta\xi(1-p-d); \text{ and}$$
$$\pi_{NN}(\xi, d_o, d_r) = 0.$$

Individuals would prefer being owner participants (OP) to being owner non-participants (ON) if and only if their extra wear and tear cost $d_o \in [0, (1-\gamma)p]$. Similarly, individuals would prefer being non-owner participants (NP) to being non-owner non-participants (NN) if and only if their inconvenience cost $d_r \in [0, 1-p]$. Therefore, in the presence of collaborative consumption, (i) individuals with $(d_o, d_r) \in [0, (1-\gamma)p] \times [0, 1-p]$ would choose strategy OP or NP; (ii) those with $(d_o, d_r) \in [0, (1-\gamma)p] \times (1-p, 1]$ would choose strategy OP or NN;

(iii) those with $(d_o, d_r) \in ((1-\gamma)p, 1] \times [0, 1-p]$ would choose strategy ON or NP; and (iv) those with $(d_o, d_r) \in ((1-\gamma)p, 1] \times (1-p, 1]$ would choose strategy ON or NN. In each case, there exists a threshold $\theta(d_o, d_r)$ such that individuals with $\xi \in [\theta(d_o, d_r), 1]$ become owners, and those with $\xi \in [0, \theta(d_o, d_r))$ become non-owners.

The threshold $\theta(d_o, d_r)$ can be expressed as follows

$$\theta(d_o,d_r) = \begin{cases} \min\left\{\left(\frac{c-((1-\gamma)p-d_o)\alpha}{p+d_r+(1-p-d_r)\alpha-((1-\gamma)p-d_o)\alpha}\right)^+, 1\right\} & \text{if } (d_o,d_r) \in [0,(1-\gamma)p] \times [0,1-p]; \\ \min\left\{\left(\frac{c-((1-\gamma)p-d_o)\alpha}{1-((1-\gamma)p-d_o)\alpha}\right)^+, 1\right\} & \text{if } (d_o,d_r) \in [0,(1-\gamma)p] \times (1-p,1]; \\ \min\left\{\left(\frac{c}{p+d_r+(1-p-d_r)\alpha}\right)^+, 1\right\} & \text{if } (d_o,d_r) \in ((1-\gamma)p,1] \times [0,1-p]; \\ c & \text{if } (d_o,d_r) \in ((1-\gamma)p,1] \times (1-p,1]. \end{cases} \tag{2.27}$$

The resulting demand (generated from non-owner participants) and supply (generated from owner participants) are respectively given by

$$D(\theta(d_o, d_r)) = \int_{\{(\xi,d_o,d_r) \in [0,\theta(d_o,d_r)) \times [0,1] \times [0,1-p]\}} \xi f(\xi, d_o, d_r),$$

and

$$S(\theta(d_o, d_r)) = \int_{\{(\xi,d_o,d_r) \in [\theta(d_o,d_r),1] \times [0,(1-\gamma)p] \times [0,1]\}} (1-\xi) f(\xi, d_o, d_r).$$

This leads to

$$\alpha = \frac{D(\theta(d_o, d_r))}{S(\theta(d_o, d_r)) + D(\theta(d_o, d_r))}. \tag{2.28}$$

An equilibrium under collaborative consumption exists if there exist $\theta^*(d_o, d_r) : [0,1]^2 \to [0,1]$ and $\alpha^* \in (0,1)$ such that (2.27) and (2.28) are satisfied.

In the following theorem, we show that an equilibrium under collaborative consumption continues to exist and is unique.

Theorem 2.13. *A unique equilibrium* $(\theta^*(d_o, d_r), \alpha^*)$ *under collaborative consumption exists for* $p \in (0, 1)$*. In equilibrium, (i) individuals with* $(\xi, d_o, d_r) \in [\theta^*(d_o, d_r), 1] \times [0, (1-\gamma)p] \times [0, 1]$ *become owner participants (OP); (ii) those with* $(\xi, d_o, d_r) \in [\theta^*(d_o, d_r), 1] \times ((1-\gamma)p, 1] \times [0, 1]$ *become owner non-participants (ON); (iii) those with* $(\xi, d_o, d_r) \in [0, \theta^*(d_o, d_r)) \times [0, 1] \times [0, 1-p]$ *become non-owner participants (NP); and (iv) those with* $(\xi, d_o, d_r) \in [0, \theta^*(d_o, d_r)) \times [0, 1] \times (1-p, 1]$ *become non-owner non-participants (NN).*

Proof. It is clear from (2.27) that $\theta(d_o, d_r)$ is decreasing in α for each $(d_o, d_r) \in [0, 1]^2$. As a result, the aggragate demand $D(\theta(d_o, d_r))$ and supply $S(\theta(d_o, d_r))$ are respectively decreasing and increasing in α. Therefore, If we let $h(\alpha) = \alpha - \frac{D(\theta(d_o,d_r))}{S(\theta(d_o,d_r))+D(\theta(d_o,d_r))}$, then h is strictly increasing in α. It is easy to see that, when $\alpha = 0$,

$$D(\theta(d_o, d_r)) = \int_{\{(\xi,d_o,d_r)\in[0,\min\{\frac{c}{p+d_r},1\})\times[0,1]\times[0,1-p]\}} \xi f(\xi, d_o, d_r) > 0,$$

and that, when $\alpha = 1$,

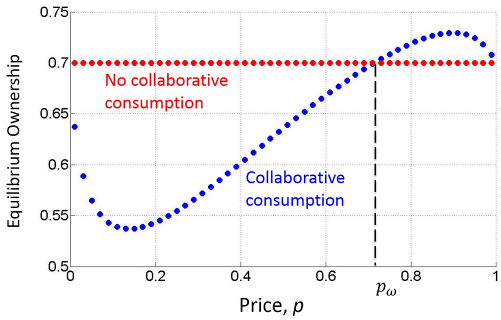
$$S(\theta(d_o, d_r)) = \int_{\{(\xi,d_o,d_r)\in[\left(\frac{c-((1-\gamma)p-d_o)}{1-((1-\gamma)p-d_o)}\right)^+,1]\times[0,(1-\gamma)p]\times[0,1]\}} (1-\xi) f(\xi, d_o, d_r) > 0.$$

So, we conclude $h(0) < 0$ and $h(1) > 0$. It follows that there exists a unique $\alpha^* \in (0, 1)$ that satisfies (2.28). The corresponding $\theta^*(w, d)$ is obtained by replacing α by α^* in (2.27). □

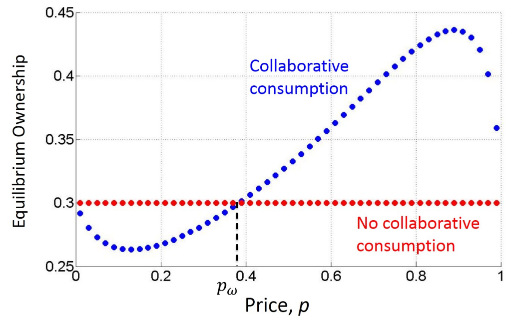
Theorem 2.13 indicates that individuals with high usage ($\xi \geq \theta^*(d_o, d_r)$) choose to be owners, among whom those who participate in collaborative consumption are less sensitive to extra wear and tear cost ($d_o \leq (1-\gamma)p$), and those who do not are more sensitive ($d_o > (1-\gamma)p$). Similarly, individuals with low usage ($\xi < \theta^*(d_o, d_r)$) choose to be non-owners, among whom those who participate have low inconvenience cost ($d_r \leq 1-p$), and those who do not have high inconvenience cost ($d_r > 1-p$).

As shown in Figure 2.11, collaborative consumption can still lead to either higher or lower ownership/usage. In particular, as in the basic model, there is a threshold for ownership/usage in the rental price above which ownership/usage is higher and below which it is lower. The threshold for ownership is again decreasing in the cost of ownership, confirming that collaborative consumption is more likely to lead to higher ownership when the cost of ownership is higher.

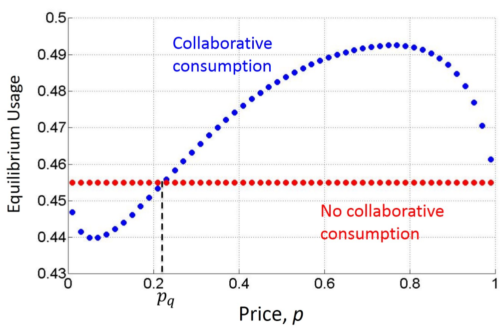
Note that, in contrast to the basic model, collaborative consumption is now possible over the full range of rental prices, including extreme values of both low and high pricing. This is because there are always individuals with sufficiently low sensitivity to extra wear and tear and inconvenience costs who would participate. More importantly, the impact of the rental price on ownership and usage are no longer monotonic. When price is moderate, ownership and usage (consistent with the basic model) are monotonically increasing in price. However, when the rental price is very high, the lack of non-owner participants (NP) eventually drives ownership and usage down to the levels of no collaborative consumption. Similarly, when the rental price is very low, the lack of owner participants (OP) eventually drives ownership and usage up to the levels of no collaborative consumption. As a result, as price increases, ownership and usage levels first decrease, then increase, before they decrease again.



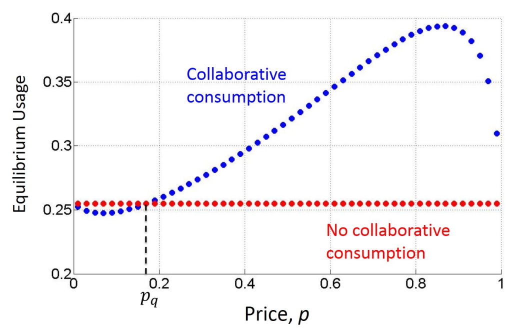
(a) $c = 0.3$, $\xi \sim U(0, 1)$, $d_o, d_r \sim Beta(1, 10)$



(b) $c = 0.7$, $\xi \sim U(0, 1)$, $d_o, d_r \sim Beta(1, 10)$



(c) $c = 0.3$, $\xi \sim U(0, 1)$, $d_o, d_r \sim Beta(1, 10)$



(d) $c = 0.7$, $\xi \sim U(0, 1)$, $d_o, d_r \sim Beta(1, 10)$

Figure 2.11: Impact of price on ownership and usage

2.6.4 Systems with Endogenous Usage

In this section, individuals are now no longer differentiated by their usage level but by their usage *valuation*, which we denote by η and assume to follow a probability distribution with a density function $f(\eta)$. We also assume that the utility derived by individuals with valuation η is linearly increasing in η and concave increasing in the usage level ξ. Specifically, the utility derived by an individual with valuation η who chooses usage level ξ is given by

$$u(\eta, \xi) = \eta\xi - \frac{1}{2}\xi^2.$$

Under this utility function, individuals with usage valuation η maximize their utility at $\xi = \eta$. Hence, the optimal usage is increasing in usage valuation, where individuals with $\eta = 0$ choose no usage and those with $\eta = 1$ choose full usage.

In the absence of collaborative consumption, individuals decide on whether or not to be owners and on their usage levels. In the presence of collaborative consumption, individuals decide in addition on whether or not to participate in collaborative consumption. Therefore, as in the previous section, owners can be classified into owner participants (OP) and owner non-participants (ON) while non-owners can be classified into non-owner participants (NP) and non-owner non-participants (NN). The payoffs associated with each of the strategies for an individual with valuation η and usage ξ are given by

$$\begin{aligned}
&\pi_{OP}(\eta,\xi) = \eta\xi - \frac{1}{2}\xi^2 + \alpha(1-\xi)((1-\gamma)p - d_o) - c;\\
&\pi_{ON}(\eta,\xi) = \eta\xi - \frac{1}{2}\xi^2 - c;\\
&\pi_{NP}(\eta,\xi) = \beta(\eta\xi - \frac{1}{2}\xi^2 - (p+d_r)\xi);\ \text{and}\\
&\pi_{NN}(\eta,\xi) = 0.
\end{aligned}$$

Then, it is easy to see that owners would participate in collaborative consumption if $p > \frac{d_o}{1-\gamma}$, they are indifferent if $p = \frac{d_o}{1-\gamma}$, and they would not participate otherwise. Therefore, for collaborative consumption to take place, we must have $p > \frac{d_o}{1-\gamma}$.

Given that $p > \frac{d_o}{1-\gamma}$, the optimal usage levels are respectively given by $\xi^*_{OP}(\eta) = (\eta - \alpha((1-\gamma)p - d_o))^+$; $\xi^*_{ON}(\eta) = \eta$; $\xi^*_{NP}(\eta) = (\eta - p - d_r)^+$; and $\xi^*_{NN}(\eta) = 0$.

Consequently, the corresponding optimal payoffs are respectively given by

$$\pi^*_{OP}(\eta) = \begin{cases} \frac{1}{2}(\eta - \alpha((1-\gamma)p - d_o))^2 - (c - \alpha((1-\gamma)p - d_o)) & \text{if } \eta > \alpha((1-\gamma)p - d_o) \\ -(c - \alpha((1-\gamma)p - d_o)) & \text{otherwise;} \end{cases}$$
$$\pi^*_{ON}(\eta) = \frac{1}{2}\eta^2 - c;$$
$$\pi^*_{NP}(\eta) = \begin{cases} \frac{1}{2}\beta(\eta - p - d_r)^2 & \text{if } \eta > p + d_r \\ 0 & \text{otherwise;} \end{cases}$$
$$\pi^*_{NN}(\eta) = 0.$$

It is easy to see that $\pi^*_{NP}(\eta) > \pi^*_{NN}(\eta)$ if and only if $\eta > p + d_r$, and $\pi^*_{OP}(\eta) - \pi^*_{NP}(\eta)$ is monotonically increasing for $\eta \in [p + d_r, 1]$. Therefore, collaborative consumption would take place if there exists $\theta \in (p + d_r, 1)$ such that

$$\pi^*_{OP}(\theta, \alpha) = \pi^*_{NP}(\theta, \alpha). \tag{2.29}$$

If such a θ exists, individuals with $\eta \in [\theta, 1]$ would become owner participants, those with $\eta \in [p + d_r, \theta)$ would become non-owner participants, and those with $\eta \in [0, p + d_r)$ would become non-owner non-participants. The aggregate demand and supply can then be expressed as

$$D(\theta, \alpha) = \int_{[p+d_r,\theta)} (\eta - p - d_r) f(\eta)\, d\eta,$$

and

$$S(\theta, \alpha) = \int_{[\theta,1]} (1 - \eta + \alpha((1-\gamma)p - d_o)) f(\eta)\, d\eta.$$

This leads to

$$\alpha = \frac{D(\theta, \alpha)}{S(\theta, \alpha) + D(\theta, \alpha)}. \tag{2.30}$$

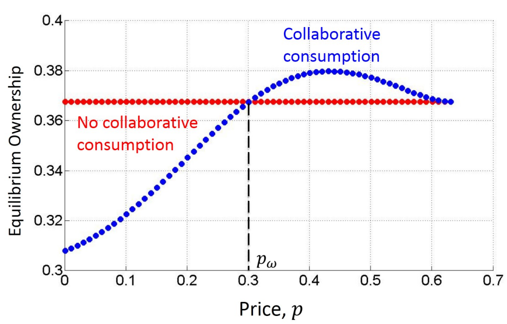
An equilibrium under collaborative consumption exists if there exists $(\theta^*, \alpha^*) \in (p+d, 1) \times (0,1)$ that is solution to (2.29) and (2.30).

In the theorem below, we provide necessary and sufficient conditions for the existence and uniqueness of the equilibrium under collaborative consumption.

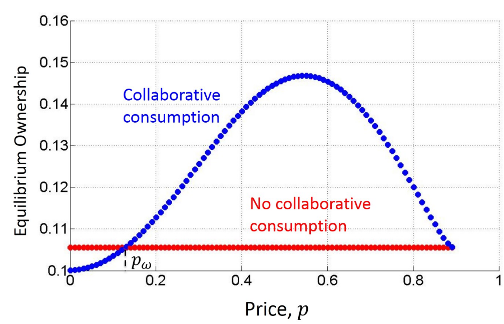
Theorem 2.14. *An equilibrium (θ^*, α^*) under collaborative consumption exists and is unique if and only if $p \in (\frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)}, \sqrt{2c} - d_r)$, in which case, individuals with $\eta \in [\theta^*, 1]$ are owner participants (OP), those with $\eta \in [p + d_r, \theta^*)$ are non-owner participants (NP), and those with $\eta \in [0, p + d_r)$ are non-owner non-participants (NN).*

Theorem 2.14 indicates that collaborative consumption would take place if and only if $p \in (\frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)}, \sqrt{2c} - d_r)$. This condition reduces to (i) $p \in (\frac{d_o}{(1-\gamma)}, \sqrt{2c} - d_r)$ if $c < \frac{1}{2}$, and (ii) $p \in (\frac{\sqrt{(2c-1)}+d_o}{(1-\gamma)}, 1 - d)$ if $c \geq \frac{1}{2}$. Scenario (i) corresponds to a setting where, in the absence of collaborative consumption, products are affordable to some. In this case, in the presence of collaborative consumption, if the rental price is too high ($p \geq \sqrt{2c} - d_r$), individuals prefer owning to renting. On the other hand, scenario (ii) corresponds to a setting where, in the absence of collaborative consumption, there are no owners. In this case, in the presence of collaborative consumption, if the rental price is too low ($p \leq \frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)}$), products would still be too expensive to own. In equilibrium, the population is segmented into OP, NP and NN. As rental price p increases, fewer individuals are willing to participate as non-owners. Consequently, the population of NP decreases while the population of NN increases.

Because usage is endogenous, ownership ω^* and usage q^* are no longer monotonic in the rental price p, with both first increasing and then decreasing. This is illustrated in Figure 2.12. This effect can be explained as follows. When the rental price initially increases, ownership increases because the rental income effect dominates. However, further increases in the rental price eventually makes ownership less desirable because of the diminishing demand from non-owner participants (those who decide not to own choose lower levels of usage, with this usage approaching zero as price approaches the maximum feasible value of $1 - d_r$). The lack of demand from non-owner participants eventually drives ownership and usage back down to the same levels as those in the absence of collaborative consumption.



(a) $c = 0.2$, $\gamma = 0.2$, $d_o = 0$, $d_r = 0$



(b) $c = 0.4$, $\gamma = 0.2$, $d_o = 0$, $d_r = 0$

Figure 2.12: Impact of price on ownership

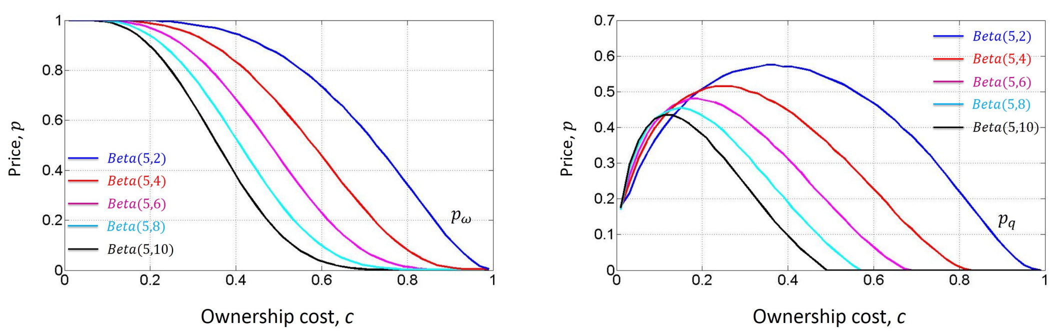
As shown in Figure 2.12, collaborative consumption can still lead to either higher or lower ownership. There is again a threshold on the rental price, p_ω, above which ownership is higher (and below which it is lower). This threshold is again decreasing in the cost of ownership, confirming that collaborative consumption is more likely to lead to higher ownership when the cost of ownership is higher.

2.6.5 Systems with General Usage Distribution

In this section, we consider the setting where individual usage follows a general distribution. We can show that Theorem 2.1 continues to hold if the usage distribution has a density function f and f is continuous with $f(\xi) > 0$ for $\xi \in (0, 1)$. (See the proof of Theorem 2.1.) In particular, ownership and usage are still strictly decreasing in the cost of ownership c, commission γ, and extra wear and tear cost d_o, and strictly increasing in the rental price p and inconvenience cost d_r. As a result, there still exist thresholds $p_q < p_\omega$ such that (i) ownership and usage are both lower if $p \leq p_q$; (ii) ownership is lower but usage is higher if $p_q < p \leq p_\omega$; and (iii) ownership and usage are both higher if $p > p_\omega$. Moreover, p_ω remains decreasing and p_q remains first increasing then decreasing in the cost of ownership, indicating that the insights developed from uniform usage distribution are still valid for general usage distributions.

Of additional interest is the impact of usage distribution on ownership and usage. We observe that the price threshold p_ω increases as usage distribution stochastically increases (i.e., there is more mass on heavy usage). This implies collaborative consumption is more likely to lead to lower ownership when usage distribution is stochastically larger. This effect appears to be related to the fact that p_ω is decreasing in the cost of ownership (as shown in Proposition 3). When usage distribution is stochastically larger, there are more owners in the absence of collaborative consumption (which is similar to the case where the cost of ownership is lower). Introducing peer-to-peer product sharing in this case benefits renters more, for the probability of finding an available product is higher. Therefore, price has to be higher for collaborative consumption to lead to higher ownership. Similarly, p_q is observed to be mostly increasing in usage distribution (except when the cost of ownership is relatively low), implying that collaborative consumption is also more likely to lead to lower usage when usage distribution is stochastically larger.

Figure 2.13 illustrates the price thresholds p_ω and p_q for $Beta(a, b)$ distributions with $a = 5$ and $b = 2, 4 \ldots, 10$ (a smaller b indicates a stochastically larger usage distribution). As shown in Figure 2.13, p_ω and p_q are mostly increasing in usage distribution.



(a) $\gamma = 0.2$, $d_o = 0$, $d_r = 0$ (b) $\gamma = 0.2$, $d_o = 0$, $d_r = 0$

Figure 2.13: The impact of usage distribution on ownership and usage

2.A Proofs

Proof of Proposition 2.7: We show v_r is strictly quasiconcave in θ. This would imply v_r is also strictly quasiconcave in p, for the composition of a quasiconcave function and a monotone function is still quasiconcave. To maximize total revenue, the platform has to balance rental price p and the amount of successful transactions $\alpha(1-\theta)^2/2$. Among the duo, rental price p is a decreasing function in θ, while the amount of transaction $\alpha(1-\theta)^2/2$ is a quasiconcave function in θ that peaks at $\theta = \frac{1}{2}$, the point where rental supply equals rental demand. This implies $v_r(\theta)$ in (2.18) is strictly decreasing on $[\max\{\frac{1}{2}, \underline{\theta}\}, \overline{\theta}]$. Therefore, to show $v_r(\theta)$ is strictly quasiconcave on $[\underline{\theta}, \overline{\theta}]$, it suffices to show that it is strictly quasiconcave on $(0, \frac{1}{2})$.

As v_r is smooth in θ, we have

$$\frac{\partial v_r}{\partial \theta} = \frac{\gamma}{2} \frac{c(2\theta^2-2\theta+1)^2(\gamma\theta^2-2\theta+1)+\theta^3(-4\gamma\theta^4+(8\gamma+6)\theta^3-(8\gamma+12)\theta^2+(3\gamma+11)\theta-4)}{(\gamma\theta-1)^2(2\theta^2-2\theta+1)^2}. \tag{2.31}$$

Note that $\frac{\partial v_r}{\partial \theta}(0) > 0$ and $\frac{\partial v_r}{\partial \theta}(\frac{1}{2}) < 0$. We claim that the numerator is strictly decreasing in $\theta \in (0, \frac{1}{2})$. Suppose this is true. Then, $\frac{\partial v_r}{\partial \theta}$ is first positive then negative on $(0, \frac{1}{2})$, and therefore v_r is strictly quasiconcave on $(0, \frac{1}{2})$. To show the claim is true, we decompose the numerator of (2.31) into $g_1(\theta, \gamma) = (2\theta^2 - 2\theta + 1)^2(\gamma\theta^2-2\theta+1)$, and $g_2(\theta, \gamma) = \theta^3(-4\gamma\theta^4+(8\gamma+6)\theta^3-(8\gamma+12)\theta^2+(3\gamma+11)\theta-4)$. It is easy to show that g_1 and g_2 are both strictly decreasing in θ. $\square$

Proof of Proposition 2.8: We first show that θ_r^* is increasing in c. Observe $\theta_r^* = \underline{\theta}$ if $\underline{\theta} \geq \frac{1}{2}$, and $\theta_r^* < \frac{1}{2}$ if $\underline{\theta} < \frac{1}{2}$. As $\underline{\theta} = \theta^*(1)$ is increasing in c, it suffices to assume $\underline{\theta} < \frac{1}{2}$ and show θ_r^* is increasing in c in this case. By (2.18), $\frac{\partial^2 \pi_p}{\partial\theta\partial c} = \frac{\gamma(\gamma\theta^2-2\theta+1)}{(\gamma\theta-1)^2} > 0$ for $\theta < \frac{1}{2}$. This implies v_r is supermodular in (θ, c) on $[0, \frac{1}{2}]^2$. By the Topkis Theorem (Topkis [1998, Lemma 2.8.1]), θ_r^* is increasing in c.

Next, we show that v_r^* is strictly quasiconcave in c. As θ in Theorem 2.1 is continuously differentiable, $\frac{\partial v_r}{\partial c}$ is continuous. By the Envelope Theorem (Milgron and Segal [2002, Corollary 4]), $\frac{\partial v_r^*}{\partial c}(\gamma, c) = \frac{\partial v_r}{\partial c}(p_r^*, \gamma, c)$ holds on any compact interval in $(0,1)$. As $(0,1)$ can be covered by an increasing union of compact subintervals, the envelope equation holds on entire $(0,1)$. Therefore, we have $\frac{\partial v_r^*}{\partial c}(\gamma, c) = \gamma p_r^* \frac{2\theta_r^*(\theta_r^*-1)(2\theta_r^*-1)(\theta_r^{*2}-\theta_r^*+1)}{(2\theta_r^{*2}-2\theta_r^*+1)^2} \frac{\partial\theta}{\partial c}(p_r^*, \gamma, c)$. Proposition 2.2 shows $\frac{\partial\theta}{\partial c} > 0$. So, $\frac{\partial v_r^*}{\partial c} > 0$ if $\theta_r^* < \frac{1}{2}$, $\frac{\partial v_r^*}{\partial c} = 0$ if $\theta_r^* = \frac{1}{2}$, and $\frac{\partial v_r^*}{\partial c} < 0$ if $\theta_r^* > \frac{1}{2}$. As θ_r^* is increasing in c, we conclude v_r^* is quasiconcave in c. Moreover, substituting θ with $\frac{1}{2}$ in (2.31) yields $\frac{\partial v_r}{\partial\theta}(\frac{1}{2}) = \frac{\gamma(2c\gamma+\gamma-3)}{4(\gamma-2)^2} < 0$. This implies $\theta_r^* = \frac{1}{2}$ iff $\underline{\theta} = \frac{1}{2}$. Therefore, v_r^* is strictly quasiconcave in c.

It remains to be shown that v_r^* has a strictly increasing as well as a strictly decreasing segment. Given what we have shown, it is sufficient to prove that, as c ranges through $(0,1)$, $\overline{\theta} = \theta(0)$ (as a function of c) has a segment below $\frac{1}{2}$, and $\underline{\theta} = \theta(1)$ has a segment above $\frac{1}{2}$. From (2.10), $\theta(0) < \frac{1}{2}$ iff $\phi(\frac{1}{2}, 0) > 0$, and $\theta(1) > \frac{1}{2}$ iff $\phi(\frac{1}{2}, 1) < 0$. The former is equivalent to $c < \frac{1}{4}$, and the latter to $c > \frac{3}{4} - \frac{1}{4}\gamma$. Therefore, v_r^* is strictly increasing when $c < \frac{1}{4}$, and strictly decreasing when $c > \frac{3}{4} - \frac{1}{4}\gamma$. As c ranges through $(0,1)$, both segments are non-empty. $\square$

Proof of Proposition 2.9: By Proposition 4, $c \in (\underline{\theta}, \overline{\theta})$ for $(\gamma, c) \in (0,1)^2$. As v_r is quasiconcave, $\omega_r^*(c) \le \hat{\omega}(c)$ iff $\theta_r^*(c) \ge c$ iff $\frac{\partial v_r}{\partial\theta}(c) \ge 0$. Replacing θ by c in

(2.31), we have

$$\frac{\partial v_r}{\partial \theta}(c) = \frac{\gamma}{2}\frac{c(1-c)(2c^4 - 6c^3 + (7+\gamma)c^2 - 5c + 1)}{(\gamma c - 1)^2(2c^2 - 2c + 1)^2}.$$

Let $g(c,\gamma) = 2c^4 - 6c^3 + (7+\gamma)c^2 - 5c + 1$. Then $\frac{\partial v_r}{\partial \theta}(c) \geq 0$ iff $g(c,\gamma) \geq 0$. Moreover, $g(0,\gamma) = 1 > 0$, $g(1,\gamma) = \gamma - 1 < 0$, and g is strictly convex in c, as $\frac{\partial^2 g}{\partial c^2} = 24c^2 - 36c + 14 + 2\gamma > 0$. It follows that $g(c,\gamma) = 0$ has a unique solution $c_{r,\omega}(\gamma) \in (0,1)$ with $\frac{\partial v_r}{\partial \theta}(c) > 0$ if $c < c_{r,\omega}(\gamma)$ and $\frac{\partial v_r}{\partial \theta}(c) < 0$ if $c > c_{r,\omega}(\gamma)$. This implies $\omega_r^*(c) \leq \hat{\omega}(c)$ if $c \leq c_{r,\omega}(\gamma)$ and $\omega_r^*(c) > \hat{\omega}(c)$ otherwise. It is easy to see that $c_{r,\omega}(\gamma)$ is strictly increasing in γ. □

Proof of Proposition 2.10: In what follows, we derive the strictly positive decreasing function $\gamma_s(c)$, and show that $\theta_s^* \in (\underline{\theta}, \overline{\theta})$ if and only if $\gamma < \gamma_s(c)$. In other words, if $\gamma < \gamma_s(c)$, θ_s^* satisfies $\frac{\partial v_s}{\partial \theta}(\theta_s^*) = 0$. As v_s is strictly concave in $[0,1]$, this implies θ_s^* is also optimal to the social planner's problem.

We denote by, θ_c^*, the unique optimal solution to $\max_{\theta \in [0,1]} v_s$ for $c \in [0,1]$. By the Maximum Theorem and Topkis Theorem, θ_c^* is continuously increasing in c. For $c \in (0,1)$, as

$$\frac{\partial v_s}{\partial \theta} = \frac{1}{4}(-1 + 4c - 2\theta + \frac{(1-2\theta)}{(2\theta - 2\theta + 1)^2}) \tag{2.32}$$

is stricly positive when $\theta = 0$, and strictly negative when $\theta = 1$, we know that v_s, apart from being concave, is first increasing, then decreasing on $[0,1]$. Therefore, $\frac{\partial v_s}{\partial \theta}(\theta_c^*) = 0$. It follows that θ_c^* satisfies $\frac{(2\theta_c^{*5} - 3\theta_c^{*4} + 2\theta_c^{*3})}{(2\theta_c^{*2} - 2\theta_c^* + 1)^2} = c$. Replacing c by $\frac{(2\theta_c^{*5} - 3\theta_c^{*4} + 2\theta_c^{*3})}{(2\theta_c^{*2} - 2\theta_c^* + 1)^2}$ in (2.17) yields $p(\theta_c^*) = \frac{\theta_c^{*2}}{(1-\gamma\theta_c^*)(2\theta_c^{*2} - 2\theta_c^* + 1)} > 0$. If $p(\theta_c^*) < 1$, then $\theta_c^* \in (\underline{\theta}, \overline{\theta})$, in which case, $\theta_s^* = \theta_c^*$. If $p(\theta_c^*) \geq 1$, then $\theta_c^* \leq \underline{\theta}$, in which case, $\theta_s^* = \underline{\theta}$.

Observe that $p(\theta_c^*) < 1$ iff $\gamma \leq \frac{(\theta_c^*-1)^2}{\theta_c^*(2\theta_c^{*2}-2\theta_c^*+1)}$. So, if we set $\gamma_s = \frac{(\theta_c^*-1)^2}{\theta_c^*(2\theta_c^{*2}-2\theta_c^*+1)}$, then $\theta_s^* \in (\underline{\theta}, \overline{\theta})$ if $\gamma < \gamma_s$, and $\theta_s^* = \underline{\theta}$ if $\gamma \geq \gamma_s$. It is easy to see that γ_s is decreasing in θ_c^*, and therefore it is decreasing in c. $\square$

Proof of Proposition 2.11: Let $g(\theta, c) = -\frac{\theta(\theta^3-2c\theta^2+2c\theta-c)}{((1-\theta)^2+\theta^2)}$. Then, $v_r(\theta, c) = \frac{\gamma(1-\theta)}{2(1-\gamma\theta)}g(\theta, c)$. Let θ_r be the unique solution to

$$\frac{\partial v_r}{\partial \theta} = \frac{\gamma(\gamma-1)}{2(1-\gamma\theta)^2}g(\theta, c) + \frac{\gamma(1-\theta)}{2(1-\gamma\theta)}D_\theta g(\theta, c) = 0$$

in $(0, 1/2)$ (which clearly exists as $\frac{\partial v_r}{\partial \theta}(0) > 0$ and $\frac{\partial v_r}{\partial \theta}(1/2) < 0$), and θ_s be the unique solution to

$$\frac{\partial v_s}{\partial \theta} = \frac{1}{4}(-1 - 2\theta + \frac{(1-2\theta)}{(2\theta^2 - 2\theta + 1)^2} + 4c) = 0$$

in $(0, 1)$ (which clearly exists as $\frac{\partial v_s}{\partial \theta}(0) > 0$ and $\frac{\partial v_s}{\partial \theta}(1) < 0$). We claim that $\theta_r \leq \theta_s$. Suppose the claim is true. Then, $\theta_r^* = \underline{\theta} \leq \theta_s^*$ whenever $\theta_r < \underline{\theta}$, $\theta_r^* = \theta_r \leq \theta_s = \theta_s^*$ whenever $\theta_r, \theta_s \in [\underline{\theta}, \overline{\theta}]$, and $\theta_r^* \leq \overline{\theta} = \theta_s^*$ whenever $\theta_s > \overline{\theta}$. In all the cases, we have $\theta_r^* \leq \theta_s^*$.

To prove the claim that $\theta_r \leq \theta_s$, observe that g is strictly concave in θ on $[0, 1]$, since $\frac{\partial^2 g}{\partial \theta^2} = -\frac{4\theta^2(1-\theta)^2(2\theta^2-2\theta+3)}{(2\theta^2-2\theta+1)^3} < 0$. Let θ_g be the unique solution to

$$\frac{\partial g}{\partial \theta} = \frac{1}{2}(-1 - 2\theta + \frac{1-2\theta}{(2\theta^2 - 2\theta + 1)^2} + 2c) = 0$$

in $(0, 1)$. We show that $\theta_r \leq \theta_g \leq \theta_s$.

To see that $\theta_r \leq \theta_g$, note that $\frac{\partial v_r}{\partial \theta}(\theta_g) = \frac{\gamma(\gamma-1)}{2(1-\gamma\theta_g)^2}g(\theta_g, c) \leq 0$. As v_r is strictly quasiconcave on $[0, 1/2]$, it must be true that $\theta_r \leq \theta_g$. To see that $\theta_g \leq \theta_s$, note that $h(\theta) = -1 - 2\theta + \frac{1-2\theta}{(2\theta^2-2\theta+1)^2}$ is strictly decreasing. So, comparing $\frac{\partial g}{\partial \theta}$ and $\frac{\partial v_s}{\partial \theta}$, it must be true that $\theta_g \leq \theta_s$. $\square$

Proof of Proposition 2.12: From (2.26), α is strictly increasing in θ on $[0,1]$, and from (2.25), θ is decreasing in α on $[0,1]$. As $\alpha(0)=0$ and $\alpha(1)=1$, a unique equilibrium exists if $\theta(0)>0$ and $\theta(1)<1$. We claim that this is indeed the case if $\tilde{\beta} < \frac{1-c}{1-\tilde{p}-\tilde{d}_r}$.

It is easy to see that $\theta(0)$ is either 1 or $\frac{c}{p+d_r}$, and therefore $\theta(0)>0$. It is also easy to verify that $\theta(1)$ is either 0 or $\frac{c-((1-\gamma)p-d_o)}{1-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)}$, and therefore $\theta(1)<1$. In this case, a unique equilibrium under collaborative consumption exists for each $(p,\gamma,c,d_o,d_r,\tilde{\beta},\tilde{p},\tilde{d}_r)\in\Omega$. This implies that

$$\theta^* = \frac{c-((1-\gamma)p-d_o)\alpha^*}{(p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha^*}$$

always holds. Consequently, we always have $c > \alpha^*((1-\gamma)p-d_o)$ and $\alpha^*((1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)) > c-p-d_r$ in equilibrium.

On the other hand, if $\tilde{\beta} \geq \frac{1-c}{1-\tilde{p}-\tilde{d}_r}$, then $\frac{c-((1-\gamma)p-d_o)}{1-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)} \geq 1$. In this case, collaborative consumption does not take place.

Let

$$\begin{aligned}
&h(\theta,\alpha,p,\gamma,c,d_o,d_r,\tilde{\beta},\tilde{p},\tilde{d}_r)\\
=&(h_1(\cdot),h_2(\cdot))\\
=&(\theta-\frac{c-((1-\gamma)p-d_o)\alpha}{(p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha},\alpha-\frac{\theta^2}{(1-\theta)^2+\theta^2}),
\end{aligned}$$

and

$$g(p,\gamma,c,d_o,d_r,\tilde{\beta},\tilde{p},\tilde{d}_r) = (g_1(\cdot),g_2(\cdot)) = (\theta^*(\cdot),\alpha^*(\cdot)).$$

Observe that h is continuous on $[0,1]^2 \times \Omega$ unless $(\alpha, p, \gamma, d_o, d_r) = (1,1,0,0,0)$ or $(\alpha, p, d_r) = (0,0,0)$, and that g is the unique solution to $h(g(\cdot), p, \gamma, c, d_o, d_r, \tilde{\beta}, \tilde{p}, \tilde{d}_r) = 0$ in $(0,1)^2$. It is easy to show that $g(\cdot)$ is continuous on Ω.

To show (θ^*, α^*) is continuously differentiable, first note that

$$D_{(\theta,\alpha)}h = \begin{pmatrix} 1 & \frac{((1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r))c-(c-p-d_r)((1-\gamma)p-d_o)}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \\ -\frac{2\theta(1-\theta)}{[(1-\theta)^2+\theta^2]^2} & 1 \end{pmatrix}.$$

Also recall that, as $\theta^* \in (0,1)$, we always have $\alpha^*((1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)) > c-p-d_r$ and $c > \alpha^*((1-\gamma)p-d_o)$. Therefore, $D_{(\theta,\alpha)}h(\theta^*, \alpha^*, p, \gamma, c, d_o, d_r, \tilde{\beta}, \tilde{p}, \tilde{d}_r)$ is always invertible. By the Implicit Function Theorem, g is continuously differentiable, and for each component x,

$$D_x g = -[D_{(\theta,\alpha)}h]^{-1} D_x h,$$

where

$$[D_{(\theta,\alpha)}h]^{-1} = \frac{1}{\det(D_{(\theta,\alpha)}h)} \begin{pmatrix} 1 & -\frac{((1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r))c-(c-p-d_r)((1-\gamma)p-d_o)}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \\ \frac{2\theta(1-\theta)}{[(1-\theta)^2+\theta^2]^2} & 1 \end{pmatrix}.$$

Calculating $D_x f$ for each component x leads to

$$D_p h = \begin{pmatrix} \frac{[c-((1-\gamma)p-d_o)](1-\alpha)+(1-\gamma)\alpha[((1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r))\alpha-(c-p-d_r)]}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \\ 0 \end{pmatrix},$$

$$D_\gamma h = \begin{pmatrix} -\frac{p\alpha[((1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r))\alpha-(c-p-d_r)]}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \\ 0 \end{pmatrix},$$

$$D_c h = \begin{pmatrix} -\frac{1}{(p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha} \\ 0 \end{pmatrix},$$

$$D_{d_o} h = \begin{pmatrix} -\frac{\alpha[((1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r))\alpha-(c-p-d_r)]}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \\ 0 \end{pmatrix},$$

$$D_{d_r} h = \begin{pmatrix} \frac{(1-\alpha)(c-((1-\gamma)p-d_o)\alpha)}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \\ 0 \end{pmatrix},$$

$$D_{\tilde{\beta}} h = \begin{pmatrix} -\frac{\alpha(1-\tilde{p}-\tilde{d}_r)(c-((1-\gamma)p-d_o)\alpha)}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \end{pmatrix},$$

and

$$D_{\tilde{p}} h = D_{\tilde{d}_r} h = \begin{pmatrix} \frac{\tilde{\beta}\alpha(c-((1-\gamma)p-d_o)\alpha)}{\left((p+d_r)+[(1-p-d_r)-\tilde{\beta}(1-\tilde{p}-\tilde{d}_r)-((1-\gamma)p-d_o)]\alpha\right)^2} \end{pmatrix}.$$

It is clear that, in equilibrium, $D_p h > 0$, $D_\gamma h < 0$, $D_c h < 0$, $D_{d_o} h < 0$, $D_{d_r} h > 0$, $D_{\tilde{\beta}} h < 0$, and $D_{\tilde{p}} h = D_{\tilde{d}} h > 0$. Therefore, we conclude that $\frac{\partial \theta^*}{\partial p} < 0$, $\frac{\partial \alpha^*}{\partial p} < 0$, $\frac{\partial \theta^*}{\partial \gamma} > 0$, $\frac{\partial \alpha^*}{\partial \gamma} > 0$, $\frac{\partial \theta^*}{\partial c} > 0$, $\frac{\partial \alpha^*}{\partial c} > 0$, $\frac{\partial \theta^*}{\partial d_o} > 0$, $\frac{\partial \alpha^*}{\partial d_o} > 0$, $\frac{\partial \theta^*}{\partial d_r} < 0$, $\frac{\partial \alpha^*}{\partial d_r} < 0$, $\frac{\partial \theta^*}{\partial \tilde{\beta}} > 0$, $\frac{\partial \alpha^*}{\partial \tilde{\beta}} > 0$, $\frac{\partial \theta^*}{\partial \tilde{p}} < 0$, $\frac{\partial \alpha^*}{\partial \tilde{p}} < 0$, $\frac{\partial \theta^*}{\partial \tilde{d}_r} < 0$, and $\frac{\partial \alpha^*}{\partial \tilde{d}_r} < 0$. □

Proof of Theorem 2.14: To prove the theorem, we need to show that there exists a unique solution (θ^*, α^*) to (2.29) and (2.30). In what follows, we tackle the two equations one at a time.

Claim 1.

$$\alpha = \frac{D(\theta, \alpha)}{S(\theta, \alpha) + D(\theta, \alpha)}$$

admits a unique solution $\alpha^*(\theta) \in [0, 1]$ *for* $\theta \in [p + d_r, 1]$. $\alpha^*(\theta)$ *is increasing, and satisfies* $\alpha^*(p + d_r) = 0$ *and* $\alpha^*(1) = 1$.

Proof. Let

$$h(\theta,\alpha)=\alpha-\frac{D(\theta,\alpha)}{S(\theta,\alpha)+D(\theta,\alpha)}.$$

Then $h:[p+d_r,1]\times[0,1]\to\mathbb{R}$ is decreasing in θ and strictly increasing in α. This implies, for any θ, there exists at most a single $\alpha^*(\theta)\in[0,1]$ such that $h(\theta,\alpha^*(\theta))=0$. It is easy to verify that the set of θ that admits $\alpha^*(\theta)$ is a closed convex set $[\theta_1,\theta_2]$, and $\alpha^*(\theta)$ is increasing in $[\theta_1,\theta_2]$. It is also easy to verify that $\alpha^*(p+d_r)=0$ and $\alpha^*(1)=1$. Therefore, $\theta_1=p+d_r$ and $\theta_2=1$. □

Claim 2. *Suppose $p\in(\frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)},\sqrt{2c}-d_r)$. Then*

$$\pi^*_{OP}(\theta,\alpha)=\pi^*_{NP}(\theta,\alpha)$$

admits a unique solution $\theta^*(\alpha)\in[p+d_r,1]$ for $\alpha\in[\alpha_1,\alpha_2]\subset[0,1]$. $\theta^*(\alpha)$ is strictly decreasing, and one of the following is true: (i). $0=\alpha_1<\alpha_2=1$; (ii). $0<\alpha_1<\alpha_2=1$ and $\theta^*(\alpha_1)=1$; (iii). $0=\alpha_1<\alpha_2<1$ and $\theta^*(\alpha_2)=p+d_r$; or (iv). $0<\alpha_1<\alpha_2<1$, $\theta^*(\alpha_1)=1$ and $\theta^*(\alpha_2)=p+d_r$.

Proof. Let $h(\theta,\alpha)=(\pi^*_{OP}-\pi^*_{NP})(\theta,\alpha)$. Then,

$$\frac{\partial h}{\partial\theta}=\alpha(\theta-((1-\gamma)p-d_o))+(1-\alpha)(p+d_r),$$

and

$$\frac{\partial h}{\partial\alpha}=((1-\gamma)p-d_o)(1-\theta+\alpha((1-\gamma)p-d_o))+\frac{1}{2}(\theta-p-d_r)^2.$$

Therefore, $h:[p+d_r,1]\times[0,1]\to\mathbb{R}$ is strictly increasing in both θ and α. This implies, for any α, there exists at most a single $\theta^*(\alpha)\in[p+d_r,1]$ such that $h(\theta^*(\alpha),\alpha)=0$. It is easy to verify that the set of α that admits $\theta^*(\alpha)$ is a closed convex set $[\alpha_1, \alpha_2]$, and $\theta^*(\alpha)$ is strictly decreasing in $[\alpha_1, \alpha_2]$. It remains to be shown that either (i), (ii), (iii), or (iv) is true.

To begin with, we solve for $\theta^*(1) = \sqrt{2(c-((1-\gamma)p-d_o))} + ((1-\gamma)p-d_o)$ and $\theta^*(0) = \frac{p+d_r}{2} + \frac{c}{p+d_r}$. Note that the assumption $\frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)} < p < \sqrt{2c} - d_r$ implies that $\theta^*(0) > p + d_r$, and $\theta^*(1) < 1$. So, if $\theta^*(0) \leq 1$ and $\theta^*(1) \geq p + d_r$, then we have case (i).

If $\theta^*(0) > 1$, then $(p+d_r)^2 - 2(p+d_r) + 2c > 0$. In this case, we have $h(1,0) = -\frac{1}{2}((p+d_r)^2 - 2(p+d_r) + 2c) < 0$, and $h(1,1) = \frac{1}{2}(1+((1-\gamma)p-d_o)^2) - c > 0$. So, there exists $\alpha_1 \in (0,1)$ such that $h(1,\alpha_1) = 0$, which also implies that $\theta^*(\alpha_1) = 1$. Similarly, if $\theta^*(1) < p+d_r$, then $\frac{1}{2}(p+d_r-((1-\gamma)p-d_o))^2-(c-((1-\gamma)p-d_o)) > 0$. In this case, we have $h(p+d_r,0) = \frac{1}{2}(p+d_r)^2 - c < 0$, and $h(p+d_r,1) = \frac{1}{2}(p+d_r-((1-\gamma)p-d_o))^2-(c-((1-\gamma)p-d_o)) > 0$. So, there exists $\alpha_2 \in (0,1)$ such that $h(p+d_r, \alpha_2) = 0$, which also implies $\theta^*(\alpha_2) = p + d_r$. It follows that if $\theta^*(0) > 1$ and $\theta^*(1) \geq p + d_r$, we have case (ii); If $\theta^*(0) \leq 1$ and $\theta^*(1) < p + d_r$, we have case (iii); If $\theta^*(0) > 1$ and $\theta^*(1) < p + d_r$, we have case (iv). We have covered all the cases. □

To finish the proof, we first show that an equilibrium under collaborative consumption does not exist for $p \leq \frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)}$ and $p \geq \sqrt{2c} - d_r$. If $p \leq \frac{d_o}{(1-\gamma)}$, then ON dominates OP for $\eta \in [0,1]$. If $\frac{d_o}{(1-\gamma)} < p \leq \frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)}$, then

$$\begin{aligned}
\pi^*_{OP}(1,\alpha) - \pi^*_{NP}(1,\alpha) &= \frac{1}{2}(1-\alpha((1-\gamma)p-d_o))^2 - (c-\alpha((1-\gamma)p-d_o)) - \pi^*_{NP}(1,\alpha) \\
&\leq \frac{1}{2}(1+\alpha^2((1-\gamma)p-d_o)^2) - c \\
&\leq 0,
\end{aligned}$$

in which case, NP dominates OP for $\eta \in [p+d_r, 1]$. On the other hand, if $p \geq \sqrt{2c} - d_r$, then

$$\begin{aligned}
&\pi^*_{OP}(p+d_r,\alpha) - \pi^*_{NP}(p+d_r,\alpha)\\
=&\frac{1}{2}((p+d_r)^2 + \alpha^2((1-\gamma)p - d_o)^2) + (1-p-d_r)\alpha((1-\gamma)p-d_o) - c\\
\geq&\frac{1}{2}(p+d_r)^2 - c\\
\geq&0,
\end{aligned}$$

in which case, OP dominates NP for $\eta \in [p+d_r, 1]$. In these cases, collaborative consumption cannot take place, and individuals become ON's and NN's.

Now, we show that a unique equilibrium exists for $p \in (\frac{\sqrt{(2c-1)^+}+d_o}{(1-\gamma)}, \sqrt{2c} - d_r)$. By Claim 1, there is a unique $\alpha^*(\theta) \in [0,1]$ for $\theta \in [p+d_r, 1]$ that is a solution to (2.30). $\alpha^*(\theta)$ is increasing, and satisfies $\alpha^*(p+d_r) = 0$ and $\alpha^*(1) = 1$. By Claim 2, there is a unique $\theta^*(\alpha) \in [p+d_r, 1]$ for $\alpha \in [\alpha_1, \alpha_2] \subset [0,1]$ that is a solution to (2.29). $\theta^*(\alpha)$ is strictly decreasing, and one of the four cases is true. Due to the monotonicity of $\alpha^*(\theta)$ and $\theta^*(\alpha)$, the functions can have at most one intersection in $[p+d, 1] \times [0,1]$. Therefore, every equilibrium is unique. It is easy to verify that, in either of the four cases, they must intersect each other at some $(\theta^*, \alpha^*) \in (p+d_r, 1) \times (0,1)$. We have proved the theorem. $\square$

2.B An Alternative Model for Matching Parameters

In this section, we describe an alternative model for the matching parameters α and β. Consider a setting similar in all aspects to the basic model described in Sections 3 and 4, except for the fact that the continuum of individuals is now approximated by a finite population, and usage is random and arises over multiple periods. To be precise, we consider a population of finitely many individuals $i \in \{1, 2, \ldots, N\}$ whose usage per period $\{X_i\}$ are independent random variables with mean $\{\xi_i\}$. For example, such a situation arises when, in each period, individual i would use the product with probability ξ_i (and, therefore, would not use the product with probability $1 - \xi_i$). In other words, usage level in each period is either 1 or 0 with probability ξ_i and $1 - \xi_i$, respectively. In this case, we have

$$X_i = \begin{cases} 1 & \text{if individual } i \text{ would like to use the product;} \\ 0 & \text{otherwise.} \end{cases}$$

In the presence of collaborative consumption, some individuals become owners and others become renters. Without loss of generality, the set of owners is assumed to be $\{1, \ldots, M\}$ and the rest ($\{M+1, \ldots, N\}$) are assumed to be renters. Then, the aggregate supply S and demand D in a given period can be expressed as $S = \sum_{i=1}^{M} X_i$ and $D = \sum_{i=M+1}^{N} X_i$. The amount of successful transactions in such a period is then given by $Z = \min\{S, D\}$.

In a setting of multiple periods (i.e., $t = 1, \ldots, T$), the matching parameters α, which represents the overall fraction of total supply that is matched to renters, would be given by $\sum_{t=1}^{T} Z_t / \sum_{t=1}^{T} S_t$. Similarly, the matching parameter β, which represents the overall fraction of total demand that is fulfilled by owners, would be given by $\sum_{t=1}^{T} Z_t / \sum_{t=1}^{T} D_t$. If individual usage is assumed to be i.i.d. over time periods, then by the Law of Large Numbers,

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} Z_t}{\sum_{t=1}^{T} S_t} \to \frac{E[Z]}{E[S]} \quad a.e.,$$

and

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} Z_t}{\sum_{t=1}^{T} D_t} \to \frac{E[Z]}{E[D]} \quad a.e.$$

Therefore, if the number of periods is sufficiently large, the matching probabilities α and β would be approximately

$$\alpha = \frac{E[Z]}{E[S]},$$

and

$$\beta = \frac{E[Z]}{E[D]}.$$

The distributions of S and D, and therefore that of Z, depend on the individual usage distributions (i.e., those of $\{X_i\}$). Consequently, as the individual usage distributions vary, the matching probabilities α and β can take various forms. For example, when the individual usage levels are distributed such that S and D can be approximated by exponential random variables, the expected transaction per period is given by

$$E[Z] = \int_{[0,\infty)} \frac{x}{E[S]} e^{-(\frac{1}{E[S]}+\frac{1}{E[D]})x}\, dx + \int_{[0,\infty)} \frac{y}{E[D]} e^{-(\frac{1}{E[S]}+\frac{1}{E[D]})y}\, dy$$
$$= \frac{\frac{1}{E[S]}}{(\frac{1}{E[S]}+\frac{1}{E[D]})^2} + \frac{\frac{1}{E[D]}}{(\frac{1}{E[S]}+\frac{1}{E[D]})^2}$$
$$= \frac{1}{\frac{1}{E[S]}+\frac{1}{E[D]}},$$

where $E[S]$ and $E[D]$ are the expectations of S and D, respectively. So, the matching probabilities would take the form of

$$\alpha = \frac{E[D]}{E[S]+E[D]},$$

and

$$\beta = \frac{E[S]}{E[S]+E[D]}.$$

We have recovered the approximations in (2.6) and (2.5). The matching friction in this case is due to the variations in supply and demand. In the case of exponentially distributed supply and demand, the coefficient of variation is 1, and the matching probabilities satisfy $\alpha + \beta = 1$. In what follows we show that the sum $\alpha + \beta$ does not necessarily have to add up to 1, and $\alpha + \beta > 1$ ($\alpha + \beta < 1$) could happen if the coefficient of variation in supply and demand is smaller (larger) than 1.

For an example of $\alpha + \beta > 1$, consider a setting where S and D can be approximated by uniformly random variables (i.e., $S \sim U[0, 2E[S]]$ and $D \sim$

$U[0, 2E[D]])$. If $E[S] \leq E[D]$, we have

$$\begin{aligned}
E[Z] &= \int_{[0,2E[S]]} x(1 - \frac{x}{2E[D]})\frac{1}{2E[S]}\,dx + \int_{[0,2E[S]]} y(1 - \frac{y}{2E[S]})\frac{1}{2E[D]}\,dy \\
&= E[S] - \frac{2E[S]^2}{3E[D]} + \frac{E[S]^2}{E[D]} - \frac{2E[S]^2}{3E[D]} \\
&= E[S] - \frac{E[S]^2}{3E[D]}.
\end{aligned}$$

In this case, α and β would take the form of

$$\alpha = 1 - \frac{E[S]}{3E[D]},$$

and

$$\beta = \frac{E[S]}{E[D]} - \frac{E[S]^2}{3E[D]^2}.$$

Similarly, if $E[S] > E[D]$, we have

$$\begin{aligned}
E[Z] &= \int_{[0,2E[D]]} x(1 - \frac{x}{2E[D]})\frac{1}{2E[S]}\,dx + \int_{[0,2E[D]]} y(1 - \frac{y}{2E[S]})\frac{1}{2E[D]}\,dy \\
&= \frac{E[D]^2}{E[S]} - \frac{2E[D]^2}{3E[S]} + E[D] - \frac{2E[D]^2}{3E[S]} \\
&= E[D] - \frac{E[D]^2}{3E[S]}.
\end{aligned}$$

In this case, α and β would take the form of

$$\alpha = \frac{E[D]}{E[S]} - \frac{E[D]^2}{3E[S]^2},$$

and

$$\beta = 1 - \frac{E[D]}{3E[S]}.$$

In both cases, the matching probabilities satisfy $\alpha + \beta > 1$. Also, note that the coefficient of variation of a $U[0,2b]$ distribution is $\frac{\sqrt{3}}{3} < 1$.

For an example of $\alpha + \beta < 1$, consider a setting where S and D can be approximated by geometric random variables (i.e., $S \sim Geo(p_1)$ and $D \sim Geo(p_2)$, where p_1 and p_2 are specified as $p_1 = \frac{1}{1+E[S]}$ and $p_2 = \frac{1}{1+E[D]}$, and where the probability mass functions are given by $P(S = x) = (1-p_1)^x p_1$ and $P(D = y) = (1-p_2)^y p_2$). Then,

$$\begin{aligned}
E[Z] &= \sum_{x=0}^{\infty} x(1-p_2)^x(1-p_1)^x p_1 + \sum_{y=0}^{\infty} y(1-p_1)^{y+1}(1-p_2)^y p_2 \\
&= \sum_{x=0}^{\infty} x((1-p_1)(1-p_2))^x p_1 + \sum_{y=0}^{\infty} y((1-p_1)(1-p_2))^y (p_2 - p_1p_2) \\
&= \frac{(1-p_1)(1-p_2)p_1}{(p_1+p_2-p_1p_2)^2} + \frac{(1-p_1)(1-p_2)(p_2-p_1p_2)}{(p_1+p_2-p_1p_2)^2} \\
&= \frac{(1-p_1)(1-p_2)}{p_1+p_2-p_1p_2}
\end{aligned}$$

It follows that α and β would take the form of

$$\alpha = \frac{p_1(1-p_2)}{p_1+p_2-p_1p_2} = \frac{E[D]}{1+E[S]+E[D]},$$

and

$$\beta = \frac{(1-p_1)p_2}{p_1+p_2-p_1p_2} = \frac{E[S]}{1+E[S]+E[D]}.$$

In this case, the matching probabilities satisfy $\alpha + \beta < 1$. Also, note that the coefficient of variation of a $Geo(p)$ distribution is $\frac{1}{\sqrt{1-p}} > 1$.

Chapter 3

Inventory Repositioning in On-Demand Product Rental Networks

3.1 Introduction

In this chapter, we consider a product rental network with a fixed number of rental units distributed across multiple locations. Inventory level is reviewed periodically and, in each period, a decision is made on how much inventory to reposition away from one location to another. Customers may pick a product up without reservation, and are allowed to keep the product for one or more periods, without committing to a specific return time or location. Thus, demand is random, and so are the rental periods and return locations of rented units. Demand that cannot be fulfilled at the location at which it arises is considered lost and incurs a lost sales penalty (or is fulfilled through other means at an additional cost). Inventory repositioning is costly and the cost depends on both the origins and destinations of the repositioning. The firm is interested in minimizing the lost revenue from unfulfilled demand (lost sales) and the cost incurred from repositioning inventory (repositioning cost). Note that more aggressive inventory repositioning can reduce lost sales but leads to higher repositioning cost. Hence, the firm must carefully mitigate the tradeoff between demand fulfillment and inventory repositioning.

Problems with the above features are many in practice. A variety of car and bike sharing programs allow customers to rent from one location and return the vehicles to another location (this option is also offered by many traditional car rental companies with customers given the option of returning a car to a different and more convenient location). In the shipping industry, shipping containers can be rented from one location and returned to a different one. These locations correspond in some cases to ports in different countries, with the need to reposition empty containers particularly acute between countries with significant trade unbalances. In large hospitals, certain medical equipment, such as IV pumps and wheelchairs, can be checked out from one location and returned to another, requiring staff to periodically rebalance this inventory among the different locations.

In this research, we are in part motivated by Redbox, a DVD rental company. Redbox rents DVD's through a network of over 40,000 kiosks in heavily trafficked areas like grocery stores and fast food restaurants. Redbox allows customers to rent from one location and return to another. According to Redbox, half of its customers take advantage of this option. The following is a quote from Matt James, Vice President of Redbox (SupplyChainBrain [2015]): "*This is an* *important part of our value proposition. We want to make it as easy as possible to rent and return a movie. But on the back end we have to deal with the problems this creates. In a network as large as Redbox's, imbalances can become extreme very quickly. Let's say two kiosks started with 15 disks of a particular DVD. After a week or two, one might be down to two disks and the other have 25 or 30 copies. The first one will probably stock out, which creates a bad customer experience, and the other will have a lot of disks sitting there and not being productive.*" To address these unbalances, Redbox has developed an information system that allows it to evaluate inventory levels of its kiosks overnight and determine which kiosks are under- and which kiosks are over-supplied. This information is then used the next day by field employees to remove inventory from some kiosks and place it in others.

We formulate the inventory repositioning problem as a Markov decision process. We show that the problem in each period is one that involves solving a convex optimization problem (and hence can be solved without resorting to an exhaustive search). More significantly, we show that the optimal policy in each period can be described in terms of two well-specified regions over the state space. If the system is in a state that falls within one region, it is optimal not to reposition any inventory (we refer to this region as the "no-repositioning" region). If the system is in a state that is outside this region, then it is optimal to reposition some inventory but only such that the system moves to a new state that is on the boundary of the no-repositioning region. Moreover, we provide a simple check for when a state is in the no-repositioning region, which also allows us to compute the optimal policy more efficiently.

One of the distinctive features of the problem considered lies in its nonlinear state update function (because of the lost sales feature). This non-linearity introduces technical difficulties in showing the convexity of the problem that must be solved in each period. To address this difficulty, we leverage the fact that the state update function is piecewise affine and derive properties for the directional derivatives of the value function. This approach has potential applicability to other systems with piecewise affine state update functions. Another distinctive feature of the problem is the multi-dimensionality of the state and action spaces. Unlike many classical inventory problems, the optimal inventory repositioning policy cannot be characterized by simple thresholds in the state space, as increasing inventory at one location requires reducing inventory at some other locations. Instead, we show that the optimal policy is defined by a no-repositioning region within which it is optimal to do nothing and outside of which it is optimal to reposition to the region's boundary. Such an optimal policy not only generalizes the threshold policy for two-location problems (i.e., it implies a simple threshold policy for two-location problems) but also preserves some of the computational benefits. Therefore, the results in this paper may also be useful in informing future studies of multi-dimensional problems.

The rest of the chapter is organized as follows. In Section 3.2, we review related literature. In Section 3.3, we describe and formulate the problem. In Section 3.4, we analyze the structure of the optimal policy for the special case of a single period problem. In Section 3.5, we use the results from the single period problem to extend the analysis to problems with finitely and infinitely many periods.

3.2 Related Literature

Our work is related to the literature on inventory transshipment, where transshipment refers to the ability to transfer inventory from one location to another (see Paterson et al. [2011] for a comprehensive review). Much of this literature focuses on reactive transshipment, where transshipment occurs in response to demand realization and transshipped inventory can be used immediately to fulfill demand. Examples include Robinson [1990], Herer et al. [2006], Hu et al. [2008], and Yao et al. [2015].

A relatively limited number of papers consider proactive transshipment, where transshipment is carried out before and, in anticipation of, demand realization. Examples include Karmarkar and Patel [1977], Karmarkar [1981], and Abouee-Mehrizi et al. [2015]. However, existing analytical results regarding optimal policies are mostly for problems with a single period. Results for multiple periods are relatively limited and focus for the most part on systems where unfulfilled demand is back-ordered. Karmarkar [1981] considers a multi-period problem where unfulfilled demand is backordered and notes the difficulty of treating lost sales. Abouee-Mehrizi et al. [2015] consider a multi-period two-location problem with lost sales and show that the optimal policy in this case can be described by multiple regions, where in each region the optimal policy takes a different form.

Our problem has the feature of proactive transshipment. However, in our case inventory cannot be replenished from an external supply source (as is the case in the transshipment literature). Instead, there is a fixed amount of inventory circulating among the different locations, and a unit that is rented is returned to a random location after a random amount of time.

Our work is also related to the literature that considers the allocation of resources in specific product rental networks, such as empty container repositioning for shipping networks and inventory repositioning in bike/car sharing systems. The literature on the former is extensive; see Lee and Meng [2015] for a comprehensive review. Most of that literature focuses on simple networks and relies on heuristics when considering more general problems; see for example Song [2005] and Li et al. [2007]. To our knowledge, there is no result regarding the optimal policy for a general network.

There is also a growing literature on inventory repositioning in bike/car sharing systems; see for example Freund et al. [2016], Shu et al. [2013], Nair and Miller-Hooks [2011], and the references therein. Most of this literature focuses on the static repositioning problem, where the objective is to find the optimal placement of bikes/cars before demand arises, with no more repositioning being made afterwards. Much of this literature employs mixed integer programming formulations and focuses on the development of algorithms and heuristics. A stream of literature models bike sharing systems as closed queueing networks and uses steady state approximations to evaluate system performance; see for example George and Xia [2011] and Fricker and Gast [2016].

Finally, our problem can be viewed as being related to the widely studied dynamic fleet management problem. The problem involves assigning vehicles to loads that originate and terminate in different locations over multiple periods. Recent examples from this literature include Topaloglu and Powell [2006], Godfrey and Powell [2002], and Powell and Carvalho [1998]. In a typical dynamic fleet management problem, movements of all vehicles, both full and empty, are decision variables. This is in contrast to our problem where the movement of vehicles is in part determined by uncontrolled events involving rentals with uncertain durations and destinations, and where decisions involve only the repositioining of unused assets. Note that most of the literature on dynamic fleet management focuses on the development of solution procedures but not on the characterization of the optimal policy.

To the best of our knowledge, our paper is the first to characterize the optimal inventory repositioning policy for a multi-period multi-location product rental network involving lost sales. Our work, by providing both analysis and solution procedures for inventory repositioning, contributes to the streams of literature where inventory repositioning is a prominent feature.

3.3 Model Description

We consider a product rental network consisting of n locations and N rental units. Inventory level is reviewed periodically and, in each period, a decision is made on how much inventory to reposition away from one location to another. Inventory repositioning is costly and the cost depends on both the origins and destinations of the repositioning. The review periods are of equal length and decisions are made over a specified planning horizon, either finite or infinite. Demand in each period is positive and random, with each unit of demand requiring the usage of one rental unit for one or more periods, with the rental period being also random. Demand that cannot be satisfied at the location at which it arises is considered lost and incurs a lost sales penalty. Units rented at one location can be returned to another. Hence, not only are rental durations random but also are return destinations. At any time, a rental unit can be either at one of the locations, available for rent, or with a customer being rented.

The sequence of events in each period is as follows. At the beginning of the period, inventory level at each location is observed. A decision is then made on how much inventory to reposition away from one location to another. Subsequently, demand is realized at each location followed by the realization of product returns.

We index the periods by $t \in \mathbb{N}$, with $t = 1$ indicating the first period in the planning horizon. We let $\boldsymbol{x}_t = (x_{t,1}, \ldots, x_{t,n})$ denote the vector of inventory levels before repositioning in period t, where $x_{t,i}$ denotes the corresponding inventory level at location i. Similarly, we let $\boldsymbol{y}_t = (y_{t,1}, \ldots, y_{t,n})$ denote the vector of inventory levels after repositioning in period t, where $y_{t,i}$ denotes the corresponding inventory level at location i. Note that inventory repositioning should always preserve the total on-hand inventory. Therefore, we require $\sum_{i=1}^{n} y_{t,i} = \sum_{i=1}^{n} x_{t,i}$.

Inventory repositioning is costly and, for each unit of inventory repositioned away from location i to location j, a cost of $c_{i,j}$ is incurred. Let $\boldsymbol{c} = (c_{i,j})$ denote the cost vector. Then, the minimum cost associated with repositioning from an inventory level $\boldsymbol{x}$ to another inventory level $\boldsymbol{y}$ is given by the following linear program.

$$
\begin{array}{llll}
\min_{z=(z_{i,j})} & \langle \boldsymbol{c}, \boldsymbol{z} \rangle & & \\
\text{subject to} & \sum_{i=1}^{n} z_{i,j} - \sum_{k=1}^{n} z_{j,k} & = y_j - x_j & \forall \quad j \\
& \boldsymbol{z} & \geq 0, &
\end{array}
\tag{3.1}
$$

where $z_{i,j}$ denotes the amount of inventory to be repositioned away from location i to location j and $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. The first constraint ensures that the change in inventory level at each location is consistent with the amounts of inventory being moved into $(\sum_i z_{i,j})$ and out of $(\sum_k z_{j,k})$ that location. The second constraint ensures that the amount of inventory being repositioned away from one location to another is always positive so that the associated cost is accounted for in the objective. Let $\boldsymbol{w} = \boldsymbol{y} - \boldsymbol{x}$. It is easy to see that the value of the linear program depends only on $\boldsymbol{w}$. Therefore, we denote the value function of (3.1) by $C(\boldsymbol{w})$ and refer to it as the inventory repositioning cost. Without loss of generality, we assume that $c_{i,j} \geq 0$ satisfy the triangle inequality (i.e., $c_{i,k} \leq c_{i,j} + c_{j,k}$ for all i, j, k).

We let $\boldsymbol{d}_t = (d_{t,1}, \ldots, d_{t,n})$ denote the vector of random demand in period t, with $d_{t,i}$ corresponding to the demand at location i. The amount of demand that cannot be fulfilled is given by $(d_{t,i} - y_{t,i})^+ = \max(0, d_{t,i} - y_{t,i})$. Let β denote the per unit lost sales penalty. Then, the total lost sales penalty incurred in period t across all locations is given by

$$L(\boldsymbol{y}_t, \boldsymbol{d}_t) = \beta \sum_{i=1}^{n} (d_{t,i} - y_{t,i})^+. \tag{3.2}$$

We assume that $\beta \geq c_{i,j}$ for $i \neq j$, that is, the cost of lost sales outweighs the cost of inventory repositioning. Under deterministic demand, this implies that it is optimal to satisfy as much demand as possible through inventory repositioning. We also assume that each product can be rented at most once within a review period, that is, rental periods are longer than review periods.

To model the randomness in both the rental periods and return locations, we assume that, at the end of each period, a random fraction $p_{t,i}$ of all the rented units $N - \sum_i (y_{t,i} - d_{t,i})^+$ is returned to location i for each $i \in \{1, \dots, n\}$, with the rest continuing to be rented. This indicates that a rented unit is memoryless about when and where it was rented, and, for a given period, all rented units have the same probability of being returned to a specific location at a specific time.

Let $\boldsymbol{p}_t$ denote the vector of random fractions $(p_{t,1}, \dots, p_{t,n})$. Then $\boldsymbol{p}_t$ must satisfy $\sum_{i=1}^n p_{t,i} \leq 1$. The case where $\sum_{i=1}^n p_{t,i} < 1$ corresponds to a setting where rental periods can be greater than one, while the case where $\sum_{i=1}^n p_{t,i} = 1$ corresponds to a setting where rental periods are equal to 1. Let μ_t denote the joint distribution of $\boldsymbol{d}_t$ and $\boldsymbol{p}_t$. We assume that the random sequence $(\boldsymbol{d}_t, \boldsymbol{p}_t)$ is independent over time, and the expected aggregate demand in each period is finite (i.e., $\int_{[0,\infty)} \sum_{i=1}^n d_{t,i} \, d\mu_t < +\infty$). We do not assume $\boldsymbol{d}_t$ and $\boldsymbol{p}_t$ to be independent. Nor do we require μ_t to have a density (or, equivalently, to be absolutely continuous).

Our modeling of unit return involves two types of memorylessness: (1) memorylessness in origin, that unit return is independent of unit origin, and (2) memorylessness in time, that unit return is independent of elapsed rental time. The first type of memorylessness is assumed for ease of exposition. Our main results continue to hold if we were to differentiate rented units by origin and associate each with a vector of random return fractions (because of the resulting cumbersome notation, we forgo this minor generality). On the other hand, the second type of memorylessness is crucial to our analysis. This assumption is justified if the rental duration can be approximated by a Geometric distribution (essentially, settings where the probability mass function for the rental duration decays exponentially over time).

The model we described can be formulated as a Markov decision process where system states correspond to the set of vectors of inventory levels $\boldsymbol{x}_t$ and the state space is specified by the n–dimensional simplex

$$S = \left\{ (x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i \leq N, x_1, \ldots, x_n \geq 0 \right\}. \tag{3.3}$$

Actions correspond to the set of vectors of inventory levels $\boldsymbol{y}_t$. Given state $\boldsymbol{x}$, the action space is specified by the $(n-1)$–dimensional simplex

$$A_I = \left\{ (y_1, \ldots, y_n) : \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i, y_1, \ldots, y_n \geq 0 \right\}, \tag{3.4}$$

where $I = \sum_i x_i$ denotes the amount of total on-hand inventory. The transition probabilities are induced by the state update function $\boldsymbol{x}_{t+1} = \tau(\boldsymbol{y}_t, \boldsymbol{d}_t, \boldsymbol{p}_t)$, where $\tau(\cdot)$ is given by

$$\tau(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p}) = (\boldsymbol{y} - \boldsymbol{d})^+ + (N - \sum_i (y_i - d_i)^+)\boldsymbol{p}. \tag{3.5}$$

Specifically, given a state $\boldsymbol{x}$ and an action $\boldsymbol{y}$, the repositioning cost is given by $C(\boldsymbol{y} - \boldsymbol{x})$, and the expected lost sales penalty is given by

$$l_t(\boldsymbol{y}) = \beta \int \sum_i (d_{t,i} - y_{t,i})^+ \, d\mu_t. \tag{3.6}$$

The cost function is the sum of the inventory repositioning cost and lost sales penalty, and is hence given by

$$r_t(\boldsymbol{x}, \boldsymbol{y}) = C(\boldsymbol{y} - \boldsymbol{x}) + l_t(\boldsymbol{y}). \tag{3.7}$$

The objective is to minimize the expected discounted cost over a specified planning horizon. In the case of a finite planning horizon with T periods, the optimality equations are given by

$$v_t(\boldsymbol{x}_t) = \min_{y_t \in A_I} \quad r_t(\boldsymbol{x}_t, \boldsymbol{y}_t) + \rho \int v_{t+1}(\boldsymbol{x}_{t+1})\, d\mu_t \tag{3.8}$$

for $t = 1, \ldots, T$, and

$$v_{T+1}(\boldsymbol{x}_{T+1}) = 0, \tag{3.9}$$

where $\rho \in [0, 1)$ is the discount factor.

It is useful to note that the problem to be solved in each period can be expressed in the following form:

$$v_t(\boldsymbol{x}_t) = \min_{y_t \in A_I} \quad C(\boldsymbol{y}_t - \boldsymbol{x}_t) + u_t(\boldsymbol{y}_t), \tag{3.10}$$

where

$$u_t(\boldsymbol{y}_t) = \int U_t(\boldsymbol{y}_t, \boldsymbol{d}_t, \boldsymbol{p}_t)\, d\mu_t, \tag{3.11}$$

and

$$U_t(\boldsymbol{y}_t, \boldsymbol{d}_t, \boldsymbol{p}_t) = L(\boldsymbol{y}_t, \boldsymbol{d}_t) + \rho v_{t+1}(\boldsymbol{x}_{t+1}). \tag{3.12}$$

Properties of v_t, u_t and U_t will be discussed in sections 3.4 and 3.5.

We conclude this section by describing properties of the repositioning cost $C(\cdot)$ defined by (3.1). These properties will be useful for characterizing the optimal policy in subsequent sections. First, the domain of $C(\cdot)$ is given by

$$dom(C) = \{\boldsymbol{w} : \boldsymbol{w} = \boldsymbol{y} - \boldsymbol{x}, \text{ where } \boldsymbol{x} \in S \text{ and } \boldsymbol{y} \in A_I\}. \tag{3.13}$$

As a linear transformation of a polyhedron, $dom(C)$ is also a polyhedron. It is easy to see that (3.1) is bounded feasible. Therefore, an optimal solution to (3.1) exists and the strong duality holds. The dual linear program can be written as follows.

$$\begin{aligned} C(\boldsymbol{w}) = \quad & \max_{\lambda=(\lambda_1,\ldots,\lambda_n)} \quad \langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle \\ & \text{subject to} \quad \lambda_j - \lambda_i \le c_{i,j} \quad \forall \quad i, j. \end{aligned} \tag{3.14}$$

Property 3.1. *$C(\cdot)$ is convex, continuous, and positively homogeneous.*

Proof. It is clear from (3.14) that $C(\cdot)$ is positively homogeneous. As the pointwise supremum of a collection of convex and lower semicontinuous functions ($\langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle$ for each $\boldsymbol{\lambda}$), C is also convex and lower semicontinuous. It is well known that a convex function on a locally simplicial convex set is upper semicontinuous (Rockafellar [1970] Theorem 10.2). Therefore, as $dom(C)$ is a polyhedron, C must be continuous. □

Due to the triangle inequality, it is not optimal to simultaneously move inventory into and out of the same location. This property can be stated as follows.

Property 3.2. *There exists an optimal solution $\boldsymbol{z}$ to (3.1) such that*

$$\sum_i z_{i,j} = (y_j - x_j)^+ \text{ and } \sum_k z_{j,k} = (y_j - x_j)^- \quad \forall \quad j.$$

Proof. It is easy to see that an equivalent condition is $z_{i,j} z_{j,k} = 0$ for all i, j, k. To show this is true, suppose $\boldsymbol{z}$ is an optimal solution and there exists i, j, k such that $z_{i,j}, z_{j,k} > 0$. If $i = k$, we can set at least one of $z_{i,j}$ and $z_{j,i}$ to 0 without violating the constraints. If $i \neq k$, we can set at least one of $z_{i,j}$ and $z_{j,k}$ to 0, and increase $z_{i,k}$ accordingly. In both cases, the resulting objective is at least as good. Repeating this for all i, k and j can enforce this condition for all i, k and j. □

Property 3.2 leads to the following bound for the repositioning cost $C(\boldsymbol{w})$.

Property 3.3.

$$C(\boldsymbol{w}) \le \frac{\beta}{2} \sum_i |w_i|. \tag{3.15}$$

Proof. By Property 3.2 and the fact that $c_{i,j} \le \beta$, there exists an optimal solution $\boldsymbol{z}$ to (3.1) such that $C(\boldsymbol{w}) = \sum_{i,j} c_{i,j} z_{i,j} = \frac{1}{2} \sum_j \sum_i c_{i,j} z_{i,j} + \frac{1}{2} \sum_j \sum_i c_{j,i} z_{j,i} \le \frac{\beta}{2} \sum_j w_j^+ + \frac{\beta}{2} \sum_j w_j^- = \frac{\beta}{2} \sum_j |w_j|$. □

It is easy to see that, in (3.15), the equality holds if $c_{i,j} = \beta$ for all i, j. Therefore, the bound is tight. In Section 3.4, we will use Property 3.3 to derive an important bound on the directional derivatives of the value function.

3.4 One-Period Problem

In this section, we study the following convex optimization problem

$$V(\boldsymbol{x}) = \min_{y \in A_I} C(\boldsymbol{y} - \boldsymbol{x}) + u(\boldsymbol{y}) \text{ for } \boldsymbol{x} \in S, \tag{3.16}$$

where S is the state space specified by (3.3), A_I is the decision space specified by (3.4), $C(\cdot)$ is the repositioning cost specified by (3.1), and $u(\cdot)$ is assumed to be a convex and continuous function that maps S into $\mathbb{R}$. In the one-period problem, $u(\cdot)$ is simply the expected lost sales penalty $l(\cdot)$. In the multi-period problem, as shown in (3.10), the problem to be solved in each period is also of the form (3.16).

3.4.1 Characterization of the Optimal Policy

The principal result of this section is the characterization of the optimal policy through the *no-repositioning set*, the collection of inventory levels from which no repositioning should be made. The *no-repositioning set* for a function $u(\cdot)$ when the total on-hand inventory level is I can be defined as follows:

$$\Omega_u(I) = \{\boldsymbol{x} \in A_I : u(\boldsymbol{x}) \leq C(\boldsymbol{y} - \boldsymbol{x}) + u(\boldsymbol{y}) \; \forall \; \boldsymbol{y} \in A_I\}. \tag{3.17}$$

By definition, no repositioning should be made from inventory levels inside $\Omega_u(I)$. In the following theorem, we show that $\Omega_u(I)$ is non-empty, connected and compact and, for inventory levels outside $\Omega_u(I)$, it is optimal to reposition to some point on the boundary of $\Omega_u(I)$. In what follows, we denote the boundary of a set E by ∂E, and the interior of E by E°.

Theorem 3.1. *The no-repositioning set $\Omega_u(I)$ is nonempty, connected and compact for all $I \in [0, N]$. An optimal policy π^* to (3.16) satisfies*

$$\begin{aligned} \pi^*(\boldsymbol{x}) &= \boldsymbol{x} && \textit{if} \quad \boldsymbol{x} \in \Omega_u(I); \\ \pi^*(\boldsymbol{x}) &\in \partial\Omega_u(I) && \textit{if} \quad \boldsymbol{x} \notin \Omega_u(I). \end{aligned} \tag{3.18}$$

Proof. Let $\boldsymbol{y}^*(\boldsymbol{x}) = \{\boldsymbol{y} \in A_I : V(\boldsymbol{x}) = C(\boldsymbol{y} - \boldsymbol{x}) + u(\boldsymbol{y})\}$ be the set of optimal solutions corresponding to the system state $\boldsymbol{x} \in S$. It is easy to verify that

$$\Omega_u(I) = \cup_{x \in A_I} \boldsymbol{y}^*(\boldsymbol{x}). \tag{3.19}$$

As $C(\cdot)$ and $u(\cdot)$ are continuous and A_I is compact, by Berge's Maximum Theorem, $\boldsymbol{y}^*(\cdot)$ is a nonempty-valued and compact-valued upper hemicontinuous[1] correspondence. As $C(\cdot)$ and $u(\cdot)$ are also convex, $\boldsymbol{y}^*(\cdot)$ is also convex-valued. So, it is clear from (3.19) that $\Omega_u(I)$ is nonempty. To show $\Omega_u(I)$ is compact,

[1]Upper hemicontinuity can be defined as follows. Suppose X and Y are topological spaces. A correspondence $f : X \rightarrow \mathcal{P}(Y)$ (power set of Y) is upper hemicontinuous if for any open set V in Y, $f^{-1}(V) = \{x \in X | f(x) \subset V\}$ is open in X.

suppose $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots$ is a sequence in $\Omega_u(I)$ such that $\boldsymbol{y}_n \in \boldsymbol{y}^*(\boldsymbol{x}_n)$ for $n \in \mathbb{N}$ and $\boldsymbol{y}_n \to \boldsymbol{y}$. We need to show that $\boldsymbol{y} \in \Omega_u(I)$. By passing through a subsequence, we may assume that $\boldsymbol{y}_{n_k} \in \boldsymbol{y}^*(\boldsymbol{x}_{n_k})$, $\boldsymbol{x}_{n_k} \to \boldsymbol{x}$ and $\boldsymbol{y}_{n_k} \to \boldsymbol{y}$. As $\boldsymbol{y}^*(\cdot)$ is compact-valued, by the Closed Graph Theorem, $\boldsymbol{y}^*(\cdot)$ has a closed graph. This implies that $\boldsymbol{y} \in \boldsymbol{y}^*(\boldsymbol{x}) \subset \Omega_u(I)$, and therefore $\Omega_u(I)$ is compact. To show that $\Omega_u(I)$ is connected, suppose the reverse is true. Then, there exist open sets V_1, V_2 in A_I such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 \supset \Omega_u(I)$, and $V_1 \cap \Omega_u(I)$ and $V_2 \cap \Omega_u(I)$ are nonempty. As $\boldsymbol{y}^*(\cdot)$ is convex-valued, this implies that, for any $\boldsymbol{x} \in A_I$, $\boldsymbol{y}^*(\boldsymbol{x})$ is either in V_1 or in V_2, but not both. Let $U_1 = \boldsymbol{y}^{*-1}(V_1)$ and $U_2 = \boldsymbol{y}^{*-1}(V_2)$. Then U_1, U_2 are open, $U_1 \cap U_2 = \emptyset$, $U_1 \cup U_2 \supset A_I$, and $U_1 \cap A_I$ and $U_2 \cap A_I$ are nonempty. This implies that the $(n-1)$–dimensional simplex A_I is not connected. We have reached a contradiction. Therefore, $\Omega_u(I)$ is also connected.

To show that π^* is optimal, note that $\pi^*(\boldsymbol{x}) = \boldsymbol{x}$ for $\boldsymbol{x} \in \Omega_u(I)$ is clear from (3.17). If $\boldsymbol{x} \notin \Omega_u(I)$, then, by (3.19), $\pi^*(\boldsymbol{x}) \in \Omega_u(I)$. Now, suppose there exists $\pi^*(\boldsymbol{x}) = \boldsymbol{y} \in \Omega_u(I)^\circ$, then $\boldsymbol{y} + t(\boldsymbol{x} - \boldsymbol{y}) \in \Omega_u(I)$ for small enough $t > 0$. Set $\boldsymbol{z} = \boldsymbol{y} + t(\boldsymbol{x} - \boldsymbol{y})$. Then $u(\boldsymbol{z}) + C(\boldsymbol{z} - \boldsymbol{x}) \le u(\boldsymbol{y}) + C(\boldsymbol{y} - \boldsymbol{z}) + C(\boldsymbol{z} - \boldsymbol{x}) = u(\boldsymbol{y}) + tC(\boldsymbol{y} - \boldsymbol{x}) + (1-t)C(\boldsymbol{y} - \boldsymbol{x}) = u(\boldsymbol{y}) + C(\boldsymbol{y} - \boldsymbol{x})$. So, $\boldsymbol{z}$ is as good a solution as $\boldsymbol{y}$. Therefore, there exists an optimal solution $\pi^*(\boldsymbol{x}) \in \partial\Omega_u(I)$ if $\boldsymbol{x} \notin \Omega_u(I)$. $\square$

3.4.2 Characterization of the No-Repositioning Set

Solving a nondifferentiable convex program such as (3.16) usually involves some computational effort. One way to reduce this effort, suggested by Theorem 3.1, is to characterize the no-repositioning set $\Omega_u(I)$. Characterizing the no-repositioning set not only solves the optimization problem for points inside $\Omega_u(I)$, but also reduces the search space to $\partial\Omega_u(I)$ for points outside $\Omega_u(I)$. To state our results, we introduce the following notation for convex functions.

Suppose $u(\cdot)$ is a convex function on $\mathbb{R}^n$ with effective domain S.[2] Let

$$u'(\boldsymbol{x};\boldsymbol{z}) = \lim_{t\downarrow 0} \frac{u(\boldsymbol{x}+t\boldsymbol{z}) - u(\boldsymbol{x})}{t} \tag{3.20}$$

denote the directional derivative of $u(\cdot)$ at $\boldsymbol{x}$ with respect to $\boldsymbol{z}$. It is well known that $u'(\boldsymbol{x};\boldsymbol{z})$ is well defined for $\boldsymbol{x} \in S$ and $\boldsymbol{z} \in \mathbb{R}^n$, and $u'(\boldsymbol{x};\cdot)$ is convex and positively homogeneous with $u'(\boldsymbol{x};0) = 0$ and $-u'(\boldsymbol{x};-\boldsymbol{z}) \le u'(\boldsymbol{x};\boldsymbol{z})$. We call $\boldsymbol{z}$ a *feasible direction* at $\boldsymbol{x}$ if $\boldsymbol{x}+t\boldsymbol{z} \in S$ for small enough $t > 0$. Of course, for $\boldsymbol{x} \in S^\circ$, all the directions are feasible. Let

$$\mathcal{H} = \{\boldsymbol{z} \in \mathbb{R}^n : \sum_i z_i = 0\}. \tag{3.21}$$

Then a feasible direction at $\boldsymbol{x}$ in $\mathcal{H}$ satisfies $\boldsymbol{x} + t\boldsymbol{z} \in A_I$ for small enough $t > 0$. In the context of our problem, a feasible direction in $\mathcal{H}$ represents an inventory repositioning that preserves the level of total on-hand inventory. By convention, $\boldsymbol{z}$ is said to be a subgradient of $u(\cdot)$ at $\boldsymbol{x}$ if $u(\boldsymbol{y}) \ge u(\boldsymbol{x}) + \langle \boldsymbol{z}, \boldsymbol{y}-\boldsymbol{x}\rangle$ for all $\boldsymbol{y}$. The set of all subgradients of $u(\cdot)$ at $\boldsymbol{x}$ is called the subdifferential of $u(\cdot)$ at $\boldsymbol{x}$ and is denoted by $\partial u(\boldsymbol{x})$. It is well known that $\partial u(\boldsymbol{x})$ is nonempty, closed and convex for $\boldsymbol{x} \in S^\circ$.

In what follows, we provide a series of first order characterizations of $\Omega_u(I)$, the first of which relies on the directional derivatives.

[2] An equivalent approach is to assume $u : S \to \mathbb{R}$ is convex and set $u(\boldsymbol{x}) = +\infty$ for $\boldsymbol{x} \notin S$.

Proposition 3.2. $\boldsymbol{x} \in \Omega_u(I)$ *if and only if*

$$u'(\boldsymbol{x};\boldsymbol{z}) \geq -C(\boldsymbol{z}) \tag{3.22}$$

for any feasible direction $\boldsymbol{z}$ *at* $\boldsymbol{x}$ *in* $\mathcal{H}$*.*

Proof. Suppose $\boldsymbol{x} \in \Omega_u(I)$. Take any feasible direction $\boldsymbol{z}$ at $\boldsymbol{x}$ in $\mathcal{H}$. Then, by (3.17),

$$\frac{u(x+tz) - u(\boldsymbol{x})}{t} \geq -C(\boldsymbol{z})$$

for $t > 0$. Taking the limit as $t \downarrow 0$, we have $u'(\boldsymbol{x};\boldsymbol{z}) \geq -C(\boldsymbol{z})$. Conversely, suppose $u'(\boldsymbol{x};\boldsymbol{z}) \geq -C(\boldsymbol{z})$ for any feasible direction $\boldsymbol{z}$ at $\boldsymbol{x}$ in $\mathcal{H}$. Let $\phi(t) = u(\boldsymbol{x}+t\boldsymbol{z})$. Then, $\phi(\cdot)$ is convex, $\phi(0) = u(\boldsymbol{x})$, and $\phi'(0+) = u'(\boldsymbol{x};\boldsymbol{z}) \geq -C(\boldsymbol{z})$. By the subgradient inequality, $t\phi'(0+) + \phi(0) \leq \phi(t)$. This implies that $-tC(\boldsymbol{z}) + u(\boldsymbol{x}) \leq u(\boldsymbol{x}+t\boldsymbol{z})$ is true for any feasible direction $\boldsymbol{z}$ at $\boldsymbol{x}$ in $\mathcal{H}$. Therefore, we have $\boldsymbol{x} \in \Omega_u(I)$. □

Proposition 3.2 is essential for several subsequent results. However, using Proposition 3.2 to verify whether a point lies inside the no-repositioning set is computationally impractical, as it involves checking an infinite number of inequalities in the form of (3.22). In the following proposition, we provide a second characterization of $\Omega_u(I)$ using the subdifferentials.

Proposition 3.3. $\boldsymbol{x} \in \Omega_u(I)$ *if*

$$\partial u(\boldsymbol{x}) \cap \mathcal{G} \neq \emptyset, \tag{3.23}$$

where $\mathcal{G} = \{(y_1,\ldots,y_n) : y_i - y_j \leq c_{ij}\ \forall\, i,j\}$*. If* $\boldsymbol{x} \in S^\circ$*, then the converse is also true.*

Proof. For the "if" part, suppose $\boldsymbol{x} \notin \Omega_u(I)$. Then, there exists $\boldsymbol{y} \in A_I$ such that $u(\boldsymbol{x}) > C(\boldsymbol{y} - \boldsymbol{x}) + u(\boldsymbol{y})$. Take any $\boldsymbol{g} \in \partial u(\boldsymbol{x})$. By the subgradient inequality, $u(\boldsymbol{x}) + \langle \boldsymbol{g}, \boldsymbol{y} - \boldsymbol{x} \rangle \leq u(\boldsymbol{y})$. It follows that

$$C(\boldsymbol{y} - \boldsymbol{x}) < -\langle \boldsymbol{g}, \boldsymbol{y} - \boldsymbol{x} \rangle.$$

Suppose $\boldsymbol{z} = (z_{ij})$ is an optimal solution to problem (3.1). Then $C(\boldsymbol{y} - \boldsymbol{x}) = \sum_i \sum_j c_{ij} z_{ij}$, and by Property 3.2, $-\langle \boldsymbol{g}, \boldsymbol{y} - \boldsymbol{x} \rangle = \sum_i g_i (y_i - x_i)^- - \sum_j g_j (y_j - x_j)^+ = \sum_i \sum_j (g_i - g_j) z_{ij}$. So, we have

$$\sum_i \sum_j c_{ij} z_{ij} < \sum_i \sum_j (g_i - g_j) z_{ij}.$$

Hence, there exists i and j such that $g_i - g_j > c_{ij}$. This implies $\boldsymbol{g} \notin \mathcal{G}$.

For the "only if" part, suppose $\boldsymbol{x} \in S^\circ$ and $\boldsymbol{x} \in \Omega_u(I)$. Assume $\partial u(\boldsymbol{x}) \cap \mathcal{G} = \emptyset$. We will show that this leads to a contradiction. Let P be the orthogonal projection from $\mathbb{R}^n$ to the subspace $\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_i x_i = 0\}$. Then

$$P(\boldsymbol{x}) = \boldsymbol{x} - \frac{\sum_i x_i}{n} e,$$

where $e = (1, \ldots, 1)$ in $\mathbb{R}^n$. Noting that $\mathcal{G} + \alpha e \subset \mathcal{G}$ for any $\alpha \in \mathbb{R}$, it is easy to verify that

$$\partial u(\boldsymbol{x}) \cap \mathcal{G} = \emptyset \text{ if and only if } P(\partial u(\boldsymbol{x})) \cap P(\mathcal{G}) = \emptyset.$$

As $P(\partial u(\boldsymbol{x}))$ is closed and $P(\mathcal{G})$ is compact, by Hahn-Banach Theorem, there exists $\boldsymbol{z} \in \mathcal{H}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that

$$\langle \boldsymbol{g}, \boldsymbol{z} \rangle < a < b < \langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle$$

for every $\boldsymbol{g} \in P(\partial u(\boldsymbol{x}))$ and for every $\boldsymbol{\lambda} \in P(\mathcal{G})$, or equivalently, as $\langle \boldsymbol{g}, \boldsymbol{z} \rangle = \langle P(\boldsymbol{g}), \boldsymbol{z} \rangle$ and $\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle = \langle P(\boldsymbol{\lambda}), \boldsymbol{z} \rangle$, for every $\boldsymbol{g} \in \partial u(\boldsymbol{x})$ and for every $\boldsymbol{\lambda} \in \mathcal{G}$. As $\boldsymbol{z}$ is a feasible direction in $\mathcal{H}$ at $\boldsymbol{x} \in \Omega_u(I)$, by Proposition 3.2, we have $u'(\boldsymbol{x}; \boldsymbol{z}) \geq -C(\boldsymbol{z})$. It follows that

$$\sup\{\langle \boldsymbol{g}, \boldsymbol{z} \rangle : \boldsymbol{g} \in \partial u(\boldsymbol{x})\} = u'(\boldsymbol{x}; \boldsymbol{z}) \geq -C(\boldsymbol{z}).$$

So, we have

$$-C(\boldsymbol{z}) \leq a < b < \langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle$$

for every $\boldsymbol{\lambda} \in \mathcal{G}$. However, by the dual formulation (3.14), there exists $\boldsymbol{\lambda} \in \{(y_1, \ldots, y_n) | y_j - y_i \leq c_{ij} \ \forall \ i, j\}$ such that $\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle = C(\boldsymbol{z})$, or equivalently, $\langle -\boldsymbol{\lambda}, \boldsymbol{z} \rangle = -C(\boldsymbol{z})$. Recognizing $-\boldsymbol{\lambda} \in \mathcal{G}$ leads to the contradiction. Therefore, $\partial u(\boldsymbol{x}) \cap \mathcal{G} \neq \emptyset$. $\square$

Proposition 3.3 suggests whether a point lies inside the no-repositioning set depends on whether $u(\cdot)$ has certain subgradients at this point. Such a characterization is useful if we can compute the subdifferential $\partial u(\boldsymbol{x})$. In particular, if $u(\cdot)$ is differentiable at $\boldsymbol{x}$, then $\partial u(\boldsymbol{x})$ consists of a single point $u'(\boldsymbol{x})$. In this case, the characterization of $\Omega_u(I)$ can be further simplified.

Corollary 3.4. *Suppose $u(\cdot)$ is differentiable at $\boldsymbol{x} \in S$. Then, $\boldsymbol{x} \in \Omega_u(I)$ if and only if*

$$\frac{\partial u}{\partial x_i}(\boldsymbol{x}) - \frac{\partial u}{\partial x_j}(\boldsymbol{x}) \leq c_{ij} \tag{3.24}$$

for all i, j.

Proof. If $u(\cdot)$ is differentiable at $\boldsymbol{x}$, then $\partial u(\boldsymbol{x}) = \{u'(\boldsymbol{x})\} = \{(\frac{\partial u}{\partial x_1}(\boldsymbol{x}), \ldots, \frac{\partial u}{\partial x_n}(\boldsymbol{x}))\}$.

In this case, it is easy to see that (3.23) is simplified to (3.24). To show that $\boldsymbol{x} \in \Omega_u(I)$ implies (3.24) for $\boldsymbol{x} \in \partial S$. Note that the equality $\sup\{\langle \boldsymbol{g}, \boldsymbol{z}\rangle : \boldsymbol{g} \in \partial u(\boldsymbol{x})\} = \langle u'(\boldsymbol{x}), \boldsymbol{z}\rangle = u'(\boldsymbol{x}; \boldsymbol{z})$ now holds for $\boldsymbol{x} \in \partial S$. The rest of the proof is the same as Proposition 3.3. □

Corollary 3.4 implies that, for a point at which $u(\cdot)$ is differentiable, determining its optimality only involves checking $n(n-1)$ inequalities. In our context, $u(\cdot)$ in the one-period problem is differentiable in S° if the demand distribution has a density. In Section 3.5, we show that the functions $(u_t(\cdot))$ in the multi-period problem are differentiable in S° if the probability distribution μ_t has a density for all t and the no-repositioning set $\Omega_u(I)$ contains no point on the boundary for $I \in (0, N)$.

The no-repositioning set $\Omega_u(I)$ can take on many forms. In what follows, we characterize the no-repositioning set for several important special cases, the first of which corresponds to a network with two locations. In this case, the no-repositioning set corresponds to a closed line segment with the boundary being the two end points. The optimal policy is a state-dependent two-threshold policy.

Example 3.1. *Suppose* $n = 2$. *By Proposition 3.2,* $\Omega_u(I) = \{(x, I - x) : x \in [s_1(I), s_2(I)]\}$ *for* $I \in [0, N]$, *where* $s_1(I) = \inf\{x : u((x, I - x); (1, -1)) \geq -c_{21}\}$ *and* $s_2(I) = \sup\{x : -u((x, I - x); (-1, 1)) \leq c_{12}\}$. *An optimal policy* π^* *to (3.16) satisfies*

$$
\begin{aligned}
\pi^*(x, I - x) &= (s_1(I), I - s_1(I)) && \text{if} \quad x < s_1(I),\\
\pi^*(x, I - x) &= (x, I - x) && \text{if} \quad s_1(I) \leq x < s_2(I),\\
\pi^*(x, I - x) &= (s_2(I), I - s_2(I)) && \text{if} \quad x \geq s_2(I).
\end{aligned}
$$

In Example 3.1, the optimal policy is described by two thresholds $s_1(I) < s_2(I)$ on the on-hand inventory level x at location 1. If x is lower than s_1, it is optimal to bring the inventory level up to s_1 by repositioning inventory from location 2 to location 1. On the other hand, if x is greater than s_2, it is optimal to bring the inventory level at location 1 down to s_2. When x falls between s_1 and s_2, it is optimal not to reposition as the benefit of inventory repositioning cannot offset the cost.

The second case corresponds to when $u(\cdot)$ is a convex quadratic function. In this case, the no-repositioning set is a polyhedron defined by $n(n-1)$ linear inequalities.

Example 3.2. *Suppose* $u(\boldsymbol{y}) = \langle \boldsymbol{B}\boldsymbol{y}, \boldsymbol{y}\rangle + \langle \boldsymbol{b}, \boldsymbol{y}\rangle + b$ *and* $\boldsymbol{B}$ *is positive semidefinite. By Corollary 3.4,* $\Omega_u(I) = \{\boldsymbol{y} \in A_I : (2\langle B_i, \boldsymbol{y}\rangle + b_i) - (2\langle B_j, \boldsymbol{y}\rangle + b_j) \leq c_{i,j}\ \forall\ i, j\}$*, where* B_i *is the* i*-th row of* $\boldsymbol{B}$*.*

An approximate solution approach can be developed based on approximating the function $u(\cdot)$ in each period by a differentiable convex function. Such an approach would benefit not only from simple characterizations of the no-repositioning set, but also from various efficient algorithms in convex optimization. In particular, in the case where $u(\cdot)$ is approximated by a convex quadratic function, the no-repositioning set is characterized as in Example 3.2 and quadratic programming can be used as the solution procedure for points lying outside the no-repositioning set. (See Keshavarz and Boyd [2014] for an example of quadratic approximate dynamic programming.)

We point out that, in general, the no-repositioning set can be non-convex.

This is illustrated in the following example.

Example 3.3. *Suppose* $u(\boldsymbol{y}) = y_1^3 + y_2^2 + y_3^2$ *and* $c_{i,j} = 0.5$. *Then, the no-repositioning set is characterized by* $\Omega_u(I) = \{\boldsymbol{y} \in A_I : -0.5 \leq 3y_1^2 - 2y_3 \leq 0.5, -0.5 \leq 3y_1^2 - 2y_2 \leq 0.5, -0.5 \leq 2y_2 - 2y_3 \leq 0.5\}$.

The no-repositioning set in Example 3.3 is not convex because the region under the parabolas $2y_2 - 3y_1^2 = 0.5$ and $2y_3 - 3y_1^2 = 0.5$ is not convex. See Figure 3.1 for the case where $I = y_1 + y_2 + y_3 = 1$.

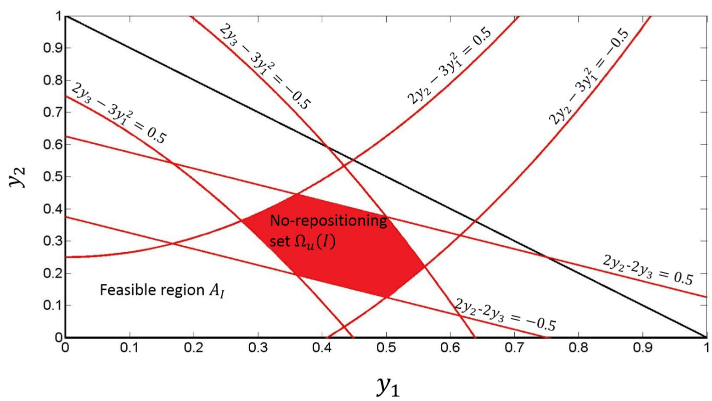


Figure 3.1: An illustration of a non-convex no-repositioning set

3.4.3 Properties of the Value Function

So far, $u(\cdot)$ is assumed to be convex and continuous in S. It turns out that, in our context, $u(\cdot)$ also satisfies the following important property:

$$u'(\boldsymbol{x};\boldsymbol{z}) \leq \beta \sum_i |z_i| \text{ for } \boldsymbol{x} \in S^\circ \text{ and } \boldsymbol{z} \in \mathbb{R}^n. \tag{3.25}$$

This property is clearly satisfied by the expected lost sales penalty $l(\cdot)$. In section 3.5, we show that the property is also satisfied by the functions $(u_t(\cdot))$ in the multi-period problem. Moreover, (3.25) implies that (i) $|u'(\boldsymbol{x};\boldsymbol{z})| \leq \beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ at $\boldsymbol{x} \in S$; (ii) $u(\cdot)$ is Lipschitz continuous; and (iii) $\partial u(\boldsymbol{x})$ is nonempty for $\boldsymbol{x} \in S$. A proof of this result can be found in Lemma 3.12 in the Appendix.

In the following lemma, we summarize the critical properties for the value function $V(\cdot)$ in our context.

Lemma 3.5. *Suppose $u'(\boldsymbol{x};\boldsymbol{z}) \leq \beta \sum_i |z_i|$ for $\boldsymbol{x} \in S^\circ$ for $\boldsymbol{z} \in \mathbb{R}^n$. Then, the value function $V(\cdot)$ is convex and continuous in S with $\Omega_V(I) = A_I$ for $I \in [0, N]$. For each $\boldsymbol{x} \in S$, the directional derivatives satisfy*

$$V'(\boldsymbol{x};\boldsymbol{z}) \leq \beta \sum_i |z_i| \tag{3.26}$$

for any feasible direction $\boldsymbol{z}$, and

$$V'(\boldsymbol{x};\boldsymbol{z}) \leq \frac{\beta}{2} \sum_i |z_i| \tag{3.27}$$

for any feasible direction $\boldsymbol{z}$ in $\mathcal{H}$.

Proof. To show that $V(\cdot)$ is convex, suppose $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are optimal solutions of (3.16) for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively. Let $I_1 = \sum_i x_{1,i}$ and $I_2 = \sum_i x_{2,i}$. Then, $\lambda \boldsymbol{y}_1 + (1-\lambda)\boldsymbol{y}_2 \in D_{\lambda I_1 + (1-\lambda) I_2}$ and $V(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leq u(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) + C(\lambda(\boldsymbol{y}_1 - \boldsymbol{x}_1) + (1-\lambda)(\boldsymbol{y}_2 - \boldsymbol{x}_2)) \leq \lambda V(\boldsymbol{x}_1) + (1-\lambda)V(\boldsymbol{x}_2)$. Continuity follows from Berge's Maximum Theorem, as the set-valued map $\boldsymbol{x} \mapsto A_I$ is continuous. Moreover, $\Omega_V(I) = A_I$ since V admits no improvement in A_I.

To show the result in (3.26), suppose $\boldsymbol{z}$ is a feasible direction at $\boldsymbol{x} \in S$. Let $\boldsymbol{y}$ be an optimal solution to (3.16) for $\boldsymbol{x}$. By Lemma 3.13 in the Appendix, there exists $\boldsymbol{w}$ such that $\sum_i |w_i| \leq \sum_i |z_i|$, $(\boldsymbol{y} + t\boldsymbol{w}) - (\boldsymbol{x} + t\boldsymbol{z}) \in dom(C)$, and $C((\boldsymbol{y}+t\boldsymbol{w})-(\boldsymbol{x}+t\boldsymbol{z})) \leq C(\boldsymbol{y}-\boldsymbol{x})$ for small enough t. So, we have $\frac{V(\boldsymbol{x}+t\boldsymbol{z})-V(\boldsymbol{x})}{t} \leq \frac{u(\boldsymbol{y}+t\boldsymbol{w})+C((\boldsymbol{y}+t\boldsymbol{w})-(\boldsymbol{x}+t\boldsymbol{z}))-u(\boldsymbol{y})-C(\boldsymbol{y}-\boldsymbol{x})}{t} \leq \frac{u(\boldsymbol{y}+t\boldsymbol{w})-u(\boldsymbol{y})}{t}$. Taking limits on both sides leads to $V'(\boldsymbol{x};\boldsymbol{z}) \leq u'(\boldsymbol{y};\boldsymbol{w}) \leq \beta \sum_i |w_i| \leq \beta \sum_i |z_i|$, where the second inequality follows from Lemma 3.12 in the Appendix.

To show the result in (3.27), suppose $\boldsymbol{z}$ is a feasible direction at $\boldsymbol{x}$ in $\mathcal{H}$. Then $\boldsymbol{y} = \boldsymbol{x}+t\boldsymbol{z} \in A_I$ for small enough $t > 0$, or equivalently, $\boldsymbol{x} = \boldsymbol{y}+s\boldsymbol{z}$ for some $s < 0$. As $V(\cdot)$ is convex, $\phi(s) = V(\boldsymbol{y} + s\boldsymbol{z})$ is convex. So, as $s < 0$, we have $V'(\boldsymbol{x};\boldsymbol{z}) = V'(\boldsymbol{y} + s\boldsymbol{z};\boldsymbol{z}) = \phi'(s+) \leq \phi'(0-) = -V'(\boldsymbol{y};-\boldsymbol{z})$. As $\Omega_V(I) = A_I$, by Proposition 3.2, $-V'(\boldsymbol{y};-\boldsymbol{z}) \leq C(-\boldsymbol{z})$. So, by Property 3.3, $V'(\boldsymbol{x};\boldsymbol{z}) \leq \frac{\beta}{2}\sum_i |z_i|$. □

The bound established in (3.26) serves as a general bound for all feasible directions, whereas the one established in (3.27), albeit sharper, is more restrictive and only applicable to feasible directions in $\mathcal{H}$. We can combine these two bounds together to form a sharper bound for feasible directions $\boldsymbol{z} \notin \mathcal{H}$. This involves decomposing $\boldsymbol{z}$ into $\boldsymbol{z}_1 + \boldsymbol{z}_2$ such that $\boldsymbol{z}_1 \in \mathcal{H}$. Such a technique proves to be useful for the multi-period analysis.

Corollary 3.6. *Suppose $V(\cdot)$ is convex and continuous in S, and $V'(\cdot;\cdot)$ satisfies (3.26) and (3.27). If $\boldsymbol{z} = \boldsymbol{z}_1+\boldsymbol{z}_2$ is a feasible direction at $\boldsymbol{x} \in S$ such that $\boldsymbol{z}_1 \in \mathcal{H}$, then $|V'(\boldsymbol{x};\boldsymbol{z})| \leq \frac{\beta}{2}\sum_i |z_{1,i}| + \beta\sum_i |z_{2,i}|$.*

Proof. For $\boldsymbol{x} \in S^\circ$, both $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are feasible directions. Therefore, for $\boldsymbol{x} \in S^\circ$, we have $V'(\boldsymbol{x};\boldsymbol{z}) \leq V'(\boldsymbol{x};\boldsymbol{z}_1) + V'(\boldsymbol{x};\boldsymbol{z}_2) \leq \frac{\beta}{2}\sum_i |z_{1,i}| + \beta\sum_i |z_{2,i}|$. The rest of the proof is similar to that of Lemma 3.12 (i) in the Appendix. □

We conclude this section with a result concerning the differentiability of $V(\cdot)$. It turns out that if $u(\cdot)$ is differentiable and the no-repositioning set contains no point on the boundary, then $V(\cdot)$ is also differentiable.

Proposition 3.7. *Suppose $\boldsymbol{y} \in S^\circ$ is an optimal solution to (3.16) for $\boldsymbol{x} \in S^\circ$. If $u(\cdot)$ is differentiable at $\boldsymbol{y}$, then $V(\cdot)$ is also differentiable at $\boldsymbol{x}$. Consequently, if $u(\cdot)$ is differentiable in S° and $\Omega_u I \subset A_I^\circ$ for $I \in (0, N)$, then $V(\cdot)$ is also differentiable in S°.*

Proof. By assumption, $V(\boldsymbol{x}) = C(\boldsymbol{y} - \boldsymbol{x}) + u(\boldsymbol{y})$ and $V(\boldsymbol{x} + t\boldsymbol{z}) \leq C(\boldsymbol{y} - \boldsymbol{x}) + u(\boldsymbol{y} + t\boldsymbol{z})$ for any direction $\boldsymbol{z}$. Therefore, we have $\frac{V(x+tz)-V(x)}{t} \leq \frac{u(y+tz)-u(y)}{t}$. Taking limits as $t \downarrow 0$ on both sides leads to $V(\boldsymbol{x}; \boldsymbol{z}) \leq u(\boldsymbol{y}; \boldsymbol{z})$. This implies that $\partial V(\boldsymbol{x}) \subset \partial u(\boldsymbol{y})$. As $\partial u(\boldsymbol{y})$ contains a single point $u'(\boldsymbol{y})$, $\partial V(\boldsymbol{x})$ also contains a single point $V'(\boldsymbol{y})$. □

3.5 Multi-Period Problem

In this section, we return to the study of the multi-period problem. The optimality equations are given by (3.8) and (3.9). It is clear from (3.11) that the problem to be solved in each period can be reduced to (3.16) with $u_t(\cdot)$ in place of $u(\cdot)$. Consequently, the optimal decision rule in each period will have the same form as the one-period problem if the functions $(u_t(\cdot))$ are convex and continuous in S.

Recall that

$$u_t(\boldsymbol{y}) = \int U_t(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p})\, d\mu_t,$$

where

$$U_t(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p}) = L_t(\boldsymbol{y}, \boldsymbol{d}) + \rho v_{t+1}(\tau(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p})).$$

If the state update function $\tau(\cdot, \boldsymbol{d}, \boldsymbol{p})$ is linear, then convexity is preserved through $v_{t+1}(\tau(\cdot, \boldsymbol{d}, \boldsymbol{p}))$. As a result, $U_{t,d,p}(\cdot) = U_t(\cdot, \boldsymbol{d}, \boldsymbol{p})$ and therefore $u_t(\cdot)$ is convex. However, with nonlinear state updates, this is not always the case. In our context, the state update function is piecewise affine, with the domain of each affine segment specified by a polyhedron. This implies that $v_{t+1}(\tau(\cdot, \boldsymbol{d}, \boldsymbol{p}))$ is not necessarily convex, but instead is piecewise convex. In spite of this, we show that $U_{t,d,p}(\cdot)$ and therefore $u_t(\cdot)$ in our context is always convex.

Lemma 3.8. *Suppose $v_{t+1}(\cdot)$ is convex and continuous in S, $v'_{t+1}(\boldsymbol{x}; \boldsymbol{z}) \leq \beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$, and $v'_{t+1}(\boldsymbol{x}; \boldsymbol{z}) \leq \frac{1}{2}\beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ in $\mathcal{H}$. Then $u_t(\cdot)$ is convex and continuous in S, and $u'_t(\boldsymbol{y}; \boldsymbol{z}) \leq \beta \sum_i |z_i|$ for $\boldsymbol{y} \in S^\circ$ for $\boldsymbol{z} \in \mathbb{R}^n$.*

Before proceeding further, we should clarify the technique that we use to prove the convexity of $U_{t,d,p}(\cdot)$. Suppose $v_{t+1}(\cdot)$ is convex and continuous. As the state update function $\tau(\cdot, \boldsymbol{d}, \boldsymbol{p})$ is piecewise affine, $U_{t,d,p}(\cdot)$ is piecewise convex, with the domain of each convex segment specified by a polyhedron $\{\boldsymbol{y} \in S : y_i \leq d_i \text{ for } i \in J, y_i \geq d_i \text{ for } i \notin J\}$ for some $J \subset \{1, \ldots, n\}$. The set of these polyhedrons forms a partition of S. Given these conditions, it can be shown that $U_{t,d,p}(\boldsymbol{y})$ is convex if and only if

$$-U_{t,d,p}(\boldsymbol{y}; -\boldsymbol{z}) \leq U_{t,d,p}(\boldsymbol{y}; \boldsymbol{z}) \tag{3.28}$$

for $\boldsymbol{y} \in S$ for $\boldsymbol{z} \in \mathbb{R}^n$. (A proof of this result can be found in Proposition 3.15 in the Appendix.) To establish the inequality (3.28), it is crucial that $v'_{t+1}(\boldsymbol{x}; \boldsymbol{z})$ satisfies the two bounds established in (3.26) and (3.27) (and therefore the statement in Corollary 3.6). For these bounds to hold for all t, we must also prove that $u_t'(\boldsymbol{y};\boldsymbol{z})$ satisfies the condition specified in (3.25) (on account of Lemma 3.5). This in turn requires us to show that (analogous to (3.25))

$$U_{t,d,p}(\boldsymbol{y};\boldsymbol{z}) \leq \beta \sum_i |z_i| \text{ for } \boldsymbol{y} \in S^\circ \text{ and } \boldsymbol{z} \in \mathbb{R}^n. \tag{3.29}$$

The proof of Lemma 3.8 is somewhat lengthy and can be found in the Appendix.

From Lemma 3.8, we are able to conclude that, for each period, the optimization problem to be solved is convex and the optimal policy is of the form (3.18). Moreover, the resulting no-repositioning sets and value functions continue to possess the properties laid out in Section 3.4.

Theorem 3.9. *The functions $(u_t(\cdot))$ are convex and continuous in S, with $u_t'(\boldsymbol{y};\boldsymbol{z}) \leq \beta \sum_i |z_i|$ for $\boldsymbol{y} \in S^\circ$ for $\boldsymbol{z} \in \mathbb{R}^n$. The no-repositioning set $\Omega_{u_t}(I)$ is nonempty, connected and compact for all $I \in [0, N]$, and can be characterized as in Proposition 3.2, 3.3 and Corollary 3.4. An optimal policy $\boldsymbol{\pi}^* = (\pi_1^*, \ldots, \pi_T^*)$ to the multi-period problem satisfies*

$$\begin{aligned} \pi_t^*(\boldsymbol{x}_t) &= \boldsymbol{x}_t && \text{if} \quad \boldsymbol{x}_t \in \Omega_{u_t}(I); \\ \pi_t^*(\boldsymbol{x}_t) &\in \partial\Omega_{u_t}(I) && \text{if} \quad \boldsymbol{x}_t \notin \Omega_{u_t}(I). \end{aligned} \tag{3.30}$$

The value functions $(v_t(\cdot))$ are convex and continuous in S, with $v_t'(\boldsymbol{x};\boldsymbol{z}) \leq \beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ and $v_t'(\boldsymbol{x};\boldsymbol{z}) \leq \frac{1}{2}\beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ in $\mathcal{H}$.

Proof. The statements for $(u_t(\cdot))$ and $(v_t(\cdot))$ follow from Lemma 3.5 and 3.8 and induction. Consequently, an optimal policy for each period is provided by Theorem 3.1, and the no-repositioning set can be characterized as in Proposition 3.2, 3.3 and Corollary 3.4. □

In what follows, we provide a result concerning the differentiability of the functions $(u_t(\cdot))$ and $(v_t(\cdot))$ in the multi-period problem.

Proposition 3.10. *Suppose μ_t is absolutely continuous and $\Omega_{u_t}(I) \subset A_I^\circ$ for $I \in (0, N)$ for each t. Then $u_t(\cdot)$ and $v_t(\cdot)$ are differentiable in S° for each t.*

Proof. Suppose $v_{t+1}(\cdot)$ is differentiable in S° and $\boldsymbol{y} \in S^\circ$. Then $U_t(\cdot, \boldsymbol{d}, \boldsymbol{p})$ is differentiable at $\boldsymbol{y}$ for almost every $(\boldsymbol{d}, \boldsymbol{p})$. (The set of points $\{(\boldsymbol{d}, \boldsymbol{p}) : d_i = y_i \text{ for some } i\}$ in which $U_t(\cdot, \boldsymbol{d}, \boldsymbol{p})$ is not differentiable in $\boldsymbol{y}$ has Lebesgue measure 0, and hence has measure 0 under μ_t.) Note that Lemma 3.12 implies that $U_t(\cdot, \boldsymbol{d}, \boldsymbol{p})$ is Lipschitz with a Lipschitz constant $n\beta$ for all $(\boldsymbol{d}, \boldsymbol{p})$. So, by the Domintated Convergence Theorem, $\frac{\partial u_t}{\partial y_i}(\boldsymbol{y}) = \int \frac{\partial U_t}{\partial y_i}(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p})\, d\mu_t$. This implies that $u_t(\cdot)$ is differentiable at $\boldsymbol{y}$. (As $u_t(\cdot)$ is convex, the partial derivatives are continuous.) Therefore, we conclude that $u_t(\cdot)$ is differentiable in S° if $v_{t+1}(\cdot)$ is differentiable in S°. The rest of the proof follows from Proposition 3.7 and induction. □

We have shown that the optimal policy for the multi-period problem has the same form as the one-period problem. In the remainder of this section, we show that the same can be said about the stationary problem with infinitely many periods. In such a problem, we denote the common distribution for $(\boldsymbol{d}_t, \boldsymbol{p}_t)$ by μ. Similarly, we denote the common values of $L_t(\cdot)$, $l_t(\cdot)$ and $r_t(\cdot)$ by $L(\cdot)$, $l(\cdot)$ and $r(\cdot)$, respectively. We use $\boldsymbol{\pi}$ to denote a stationary policy that uses the same decision rule π in each period. Under $\boldsymbol{\pi}$, the state of the process is a Markov random sequence $(X_t : t = 1, 2, \ldots)$. The optimization problem can be written as

$$V(\boldsymbol{x}) = \min_{\boldsymbol{\pi}} E_x^{\pi} \left\{ \sum_{t=1}^{\infty} \rho^{t-1} r(X_t, \pi(X_t)) \right\}, \tag{3.31}$$

where $X_1 = \boldsymbol{x}$ a.e. is the initial state of the process. Let

$$V_T(\boldsymbol{x}) = \min_{\boldsymbol{\pi}} E_x^{\pi} \left\{ \sum_{t=1}^{T} \rho^{t-1} r(X_t, \pi(X_t)) \right\} \tag{3.32}$$

denote the value function of a stationary problem with T periods. It is well known that the functions $V_T(\cdot)$ converges uniformly to $V(\cdot)$ and $V(\cdot)$ is the unique solution[3] to

$$V(\boldsymbol{x}) = \min_{y \in A_I} \quad r(\boldsymbol{x}, \boldsymbol{y}) + \rho \int V(\tau(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p}))\, d\mu. \tag{3.33}$$

Similar to the multi-period problem, the problem to be solved in each period can be reduced to the one-period problem (3.16)

$$V(\boldsymbol{x}) = \min_{y \in A_I} \quad C(\boldsymbol{y} - \boldsymbol{x}) + u(\boldsymbol{y}),$$

where $u(\boldsymbol{y}) = \int U(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p})\, d\mu$ and $U(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p}) = L(\boldsymbol{y}, \boldsymbol{d}) + \rho V(\tau(\boldsymbol{y}, \boldsymbol{d}, \boldsymbol{p}))$.

Theorem 3.11. *The function $u(\cdot)$ is convex and continuous in S, with $u'(\boldsymbol{y}; \boldsymbol{z}) \leq \beta \sum_i |z_i|$ for $\boldsymbol{y} \in S^{\circ}$ for $\boldsymbol{z} \in \mathbb{R}^n$. The no-repositioning set $\Omega_u(I)$ is nonempty, connected and compact for all $I \in [0, N]$, and can be characterized as in Proposition 3.2, 3.3 and Corollary 3.4. An optimal policy $\boldsymbol{\pi}^* = (\pi^*, \pi^*, \ldots)$ to the stationary*

[3] The reason for this, in short, is that the pointwise minimization in (3.33) defines a contraction operator on the space of bounded measurable functions on S, with $V(\cdot)$ being the unique fixed point of this operator and $V_T(\cdot)$ a Cauchy sequence that converges to $V(\cdot)$. For details, the reader may refer to Chapter 6 of Puterman [1994].

problem with infinitely many periods satisfies

$$\begin{aligned} \pi^*(\boldsymbol{x}) &= \boldsymbol{x} \quad & \textit{if} \quad \boldsymbol{x} \in \Omega_u(I); \\ \pi^*(\boldsymbol{x}) &\in \partial\Omega_u(I) \quad & \textit{if} \quad \boldsymbol{x} \notin \Omega_u(I). \end{aligned} \tag{3.34}$$

The value function $V(\cdot)$ is convex and continuous in S, with $V'(\boldsymbol{x};\boldsymbol{z}) \leq \beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ and $V'(\boldsymbol{x};\boldsymbol{z}) \leq \frac{1}{2}\beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ in $\mathcal{H}$.

Proof. As $V_T(\cdot)$ converges uniformly to $V(\cdot)$, the value function V remains to be convex and continuous. By Lemma 3.12, V_T is Lipschitz and satisfies $|V_T(\boldsymbol{x}_2) - V_T(\boldsymbol{x}_1)| \leq \beta \sum_i |x_{2,i} - x_{1,i}|$ and $|V_T(\boldsymbol{x}_2) - V_T(\boldsymbol{x}_1)| \leq \frac{\beta}{2} \sum_i |x_{2,i} - x_{1,i}|$ for $\boldsymbol{x}_2 - \boldsymbol{x}_1 \in \mathcal{H}$. Both of these inequalities are preserved by uniform convergence. It follows that $|\frac{V(x+sz)-V(x)}{s}| \leq \beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ and $|\frac{V(x+sz)-V(x)}{s}| \leq \frac{\beta}{2} \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ in $\mathcal{H}$. Therefore, the statement for $V(\cdot)$ follows from taking limits on both sides of these inequalities. The rest of the proof follows from Lemma 3.8, Theorem 3.1, Proposition 3.2, 3.3 and Corollary 3.4 $\square$

3.A Proofs

Lemma 3.12. *Suppose $u : S \to \mathbb{R}$ is convex and continuous, and $u'(\boldsymbol{x};\boldsymbol{z}) \leq \beta \sum_i |z_i|$ for $\boldsymbol{x} \in S^\circ$ for $\boldsymbol{z} \in \mathbb{R}^n$. Then, (i) $|u'(\boldsymbol{x};\boldsymbol{z})| \leq \beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ at $\boldsymbol{x} \in S$; (ii) $u(\cdot)$ is Lipschitz satisfying $|u(\boldsymbol{x}_2) - u(\boldsymbol{x}_1)| \leq \beta \sum_i |x_{2,i} - x_{1,i}|$; and (iii) $\partial u(\boldsymbol{x})$ is nonempty for $\boldsymbol{x} \in S$.*

Proof. We first show that (i) is true. Suppose $\boldsymbol{x} \in S^\circ$ and $\boldsymbol{z} \in \mathbb{R}^n$. Then, $-\beta \sum_i |z_i| \leq -u'(\boldsymbol{x};-\boldsymbol{z}) \leq u'(\boldsymbol{x};\boldsymbol{z}) \leq \beta \sum_i |z_i|$. So ,we have $|u'(\boldsymbol{x};\boldsymbol{z})| \leq \beta \sum_i |z_i|$ for $\boldsymbol{x} \in S^\circ$ for $\boldsymbol{z} \in \mathbb{R}^n$.

To show that the inequality holds for $\boldsymbol{x} \in \partial S$, first suppose $\boldsymbol{x} + s\boldsymbol{z} \in S^\circ$ for small enough $s > 0$. Let $\phi(s) = u(\boldsymbol{x} + s\boldsymbol{z})$. Then, $|\phi'(s+)| = |u'(\boldsymbol{x} + s\boldsymbol{z};\boldsymbol{z})| \leq \beta \sum_i |z_i|$ for small enough $s > 0$. As the right derivative of a continuous convex function is continuous, we have $|u'(\boldsymbol{x};\boldsymbol{z})| = |\phi'(0+)| \leq \beta \sum_i |z_i|$. Let $\boldsymbol{y} = \boldsymbol{x} + s\boldsymbol{z}$. Then, by the subgradient inequality, $u(\boldsymbol{y}) - u(\boldsymbol{x}) = \phi(s) - \phi(0) \geq s\phi'(0+) = su'(\boldsymbol{x};\boldsymbol{z})$ and, on the other hand, $u(\boldsymbol{x}) - u(\boldsymbol{y}) = \phi(0) - \phi(s) \geq -s\phi'(s-) = su'(\boldsymbol{y};-\boldsymbol{z})$. Therefore, $su'(\boldsymbol{x};\boldsymbol{z}) \leq u(\boldsymbol{y}) - u(\boldsymbol{x}) \leq -su'(\boldsymbol{y};-\boldsymbol{z})$. So, we also have $|u(\boldsymbol{x} + s\boldsymbol{z}) - u(\boldsymbol{x})| = |u(\boldsymbol{y}) - u(\boldsymbol{x})| \leq s\beta \sum_i |z_i|$.

Now, suppose $\boldsymbol{x} + s\boldsymbol{z} \in \partial S$ for small enough $s > 0$. Take $\boldsymbol{w}$ such that $\boldsymbol{x} + s(\boldsymbol{z} + \boldsymbol{w}) \in S^\circ$ for small enough $s > 0$. Then $\frac{u(\boldsymbol{x}+s\boldsymbol{z})-u(\boldsymbol{x})}{s} = \frac{u(\boldsymbol{x}+s\boldsymbol{z})-u(\boldsymbol{x}+s(\boldsymbol{z}+\boldsymbol{w}))}{s} + \frac{u(\boldsymbol{x}+s(\boldsymbol{z}+\boldsymbol{w}))-u(\boldsymbol{x})}{s}$. Taking limits on both sides yields $\liminf \frac{u(\boldsymbol{x}+s\boldsymbol{z})-u(\boldsymbol{x}+s\boldsymbol{z}+s\boldsymbol{w})}{s} + u'(\boldsymbol{x};\boldsymbol{z}+\boldsymbol{w}) \leq u'(\boldsymbol{x};\boldsymbol{z}) \leq \limsup \frac{u(\boldsymbol{x}+s\boldsymbol{z})-u(\boldsymbol{x}+s\boldsymbol{z}+s\boldsymbol{w})}{s} + u'(\boldsymbol{x};\boldsymbol{z}+\boldsymbol{w})$. So, we have $-2\beta \sum_i |w_i| - \beta \sum_i |z_i| \leq u'(\boldsymbol{x};\boldsymbol{z}) \leq 2\beta \sum_i |w_i| + \beta \sum_i |z_i|$. As $\sum_i |w_i|$ can be arbitrarily small, we must have $|u'(\boldsymbol{x};\boldsymbol{z})| \leq \beta \sum_i |z_i|$ for any feasible direction $\boldsymbol{z}$ at $\boldsymbol{x} \in S$.

To show that $u(\cdot)$ is Lipschitz, suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are in S. Let $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}$. Then, by the subgradient inequality, $u(\boldsymbol{y}) - u(\boldsymbol{x}) \geq u'(\boldsymbol{x}; \boldsymbol{z})$ and, on the other hand, $u(\boldsymbol{x}) - u(\boldsymbol{y}) \geq u'(\boldsymbol{y}; -\boldsymbol{z})$. So, we have $u'(\boldsymbol{x}; \boldsymbol{z}) \leq u(\boldsymbol{y}) - u(\boldsymbol{x}) \leq -u'(\boldsymbol{y}; -\boldsymbol{z})$, and therefore, $|u(\boldsymbol{y}) - u(\boldsymbol{x})| \leq \beta \sum_i |z_i| = \beta \sum_i |y_i - x_i|$. This implies that $u(\cdot)$ is Lipschitz (as all the norms are equivalent on finite dimensional spaces).

To show that $\partial u(\boldsymbol{x}) \neq \emptyset$, first suppose $\boldsymbol{x} \in S^\circ$. In this case, it is well known that $\partial u(\boldsymbol{x}) \neq \emptyset$. By the subgradient inequality, $\partial u(\boldsymbol{x}) \subset \Pi_i[-u'(\boldsymbol{x}; -\boldsymbol{e}_i), u'(\boldsymbol{x}; \boldsymbol{e}_i)]$, where $(\boldsymbol{e}_i : i = 1, \ldots, n)$ is the standard basis of $\mathbb{R}^n$. Therefore, by assumption, $\partial u(\boldsymbol{x}) \subset [-\beta, \beta]^n$ for $\boldsymbol{x} \in S^\circ$. This implies that $\partial u(\boldsymbol{x}) \neq \emptyset$ for $\boldsymbol{x} \in \partial S$, since the subgradient map $\boldsymbol{x} \mapsto \partial u(\boldsymbol{x})$ has a closed graph if $u(\cdot)$ is closed (See Rockafellar [1970] Theorem 24.4). □

Lemma 3.13. *Suppose $\boldsymbol{y}_0 - \boldsymbol{x}_0 \in dom(C)$ and $\boldsymbol{x}_0 + t\boldsymbol{x} \in S$ for small enough $t > 0$. Then there exists $\boldsymbol{y}$ with $\sum_i |y|_i \leq \sum_i |x|_i$ such that $(\boldsymbol{y}_0 + t\boldsymbol{y}) - (\boldsymbol{x}_0 + t\boldsymbol{x}) \in dom(C)$ and $C((\boldsymbol{y}_0 + t\boldsymbol{y}) - (\boldsymbol{x}_0 + t\boldsymbol{x})) \leq C(\boldsymbol{y}_0 - \boldsymbol{x}_0)$ for small enough t.*

Proof. It suffices to assume t is small enough such that $y_{0,j} + tx_j > 0$ for all $y_{0,j} > 0$. If $y_{0,j} + tx_j \geq 0$ for all j, then we may simply set $y = x$ and be done with it. Otherwise, there exists j such that $y_{0,j} + tx_j < 0$, or equivalently, $y_{0,j} = 0$ and $x_j < 0$. So, we assume $J = \{j : y_{0,j} = 0, x_j < 0\}$ is nonempty. Note that $J \subset \{j : y_{0,j} = 0\} \subset \{j : y_{0,j} - tx_{0,j} \leq 0\}$

Property 3.2 suggests that there exists $\boldsymbol{z}_0 \geq 0$ such that $\boldsymbol{c}\boldsymbol{z}_0 = C(\boldsymbol{y}_0 - \boldsymbol{x}_0)$ and

$$y_{0,j} - x_{0,j} = \begin{cases} -\sum_i z_{0,j,i} & \text{for} \quad j \in \{j | y_{0,j} - x_{0,j} \leq 0\}; \\ \sum_i z_{0,i,j} & \text{for} \quad j \in \{j | y_{0,j} - x_{0,j} > 0\}. \end{cases}$$

Let $\boldsymbol{y}$ be such that

$$y_j = \begin{cases} 0 & \text{for} \quad j \in J; \\ x_j & \text{for} \quad j \in \{j|y_{0,j} - x_{0,j} \leq 0\}\backslash J; \\ \sum_{i\in J} \frac{x_i}{x_{0,i}} z_{0,i,j} + x_j & \text{for} \quad j \in \{j|y_{0,j} - x_{0,j} > 0\}. \end{cases}$$

We claim that $\boldsymbol{y}$ satisfies the desired properties.

First, note that $\sum_j z_{0,i,j} = x_{0,i}$ for $i \in J$. So, we have $\sum_j |y|_j \leq \sum_{j\notin J} |x_j| + \sum_j \sum_{i\in J} \frac{z_{0,i,j}}{x_{0,i}} |x_i| = \sum_j |x|_j$. Similarly, we have $\sum_j y_j = \sum_j x_j$. Therefore, to show that $(\boldsymbol{y}_0 + t\boldsymbol{y}) - (\boldsymbol{x}_0 + t\boldsymbol{x}) \in dom(C)$, we need to verify that $y_{0,j} + ty_j \geq 0$ for all j. This is definitely true for $j \in J$. By assumption, $y_{0,j} + tx_j \geq 0$ for $j \notin J$. So, $y_{0,j} + ty_j \geq 0$ is also true for $\{j|y_{0,j} - x_{0,j} \leq 0\}\backslash J$. For $\{j|y_{0,j} - x_{0,j} > 0\}$, we have $x_{0,j} + tx_j \geq 0$ ($\frac{tx_j}{x_{0,j}} \geq -1$). It follows that $y_{0,j} - x_{0,j} + \sum_{i\in J} \frac{tx_i}{x_{0,i}} z_{0,i,j} \geq y_{0,j} - x_{0,j} - \sum_{i\in J} z_{0,i,j} \geq 0$. Summing the two inequalities up yields $y_{0,j} + \sum_{i\in J} \frac{tx_i}{x_{0,i}} z_{0,i,j} + tx_j \geq 0$.

To show that $C((\boldsymbol{y}_0 + t\boldsymbol{y}) - (\boldsymbol{x}_0 + t\boldsymbol{x})) \leq C(\boldsymbol{y}_0 - \boldsymbol{x}_0)$, let $\boldsymbol{z}$ be such that

$$z_{i,j} = \begin{cases} (1 + \frac{tx_i}{x_{0,i}}) z_{0,i,j} & \text{for} \quad i \in J; \\ z_{0,i,j} & \text{for} \quad i \notin J. \end{cases}$$

Then, we have $0 \leq z_{i,j} \leq z_{0,i,j}$ and $(y_{0,j} + ty_j) - (x_{0,j} + tx_j) = \sum_i z_{i,j} - \sum_i z_{j,i}$ for all j. (For $i \in J$, $(y_{0,i} + ty_i) - (x_{0,i} + tx_i) = y_{0,i} - x_{0,i} - tx_i = -(\sum_j z_{0,i,j} + tx_i) = -\sum_j z_{i,j}$. For $i \in \{j|y_{0,j} - x_{0,j} \leq 0\}\backslash J$, $(y_{0,i} + ty_i) - (x_{0,i} + tx_i) = y_{0,i} - x_{0,i} = -\sum_j z_{0,i,j} = -\sum_j z_{i,j}$. For $i \in \{j|y_{0,j} - x_{0,j} > 0\}$, $(y_{0,i} + ty_i) - (x_{0,i} + tx_i) = y_{0,i} - x_{0,i} + \sum_{j\in J} \frac{tx_i}{x_{0,i}} z_{0,j,i} = \sum_j z_{0,j,i} + \sum_{j\in J} \frac{tx_i}{x_{0,i}} z_{0,j,i} = \sum_j z_{j,i}$.) It follows that $\boldsymbol{z}$ is a feasible solution to (3.1) with $(\boldsymbol{y}_0 + t\boldsymbol{y})$ in place $\boldsymbol{y}$ and $(\boldsymbol{x}_0 + t\boldsymbol{x})$ in place of

$\boldsymbol{x}$. Therefore, we have $C((\boldsymbol{y}_0 + t\boldsymbol{y}) - (\boldsymbol{x}_0 + t\boldsymbol{x})) \le \boldsymbol{c}\boldsymbol{z} \le \boldsymbol{c}\boldsymbol{z}_0 = C(\boldsymbol{y}_0 - \boldsymbol{x}_0)$. The proof is complete. □

Proposition 3.14. *A function $f : (a, b) \to \mathbb{R}$ is convex if and only if it is continuous with increasing left- or right-derivative.*

Proof. The "only if" part is clear. For the "if part", we assume f is continuous with increasing right-derivative, for the proof for left-derivative is similar. It is common knowledge that a function on an open set in $\mathbb{R}^n$ is convex if and only if there exists a subgradient at every point. So, it suffices to show that $f'(x+)$ is a subgradient at x for every $x \in (a, b)$. Let $g_x(y) = f'(x+)(y - x) + f(x)$. We need to show that $f(y) \ge g_x(y)$ for $y \in (a, b)$.

We first show that $f(y) \ge g_x(y)$ if $f'(x+)$ is strictly increasing. To show this, let $h_x(y) = g_x(y) - \epsilon$ for some $\epsilon > 0$. We claim that $f(y) \ge h_x(y)$. Suppose this is not ture. Then there exists $z \in (a, b)$ such that $(f - h_x)(z) < 0$. If $z > x$, let $c = \sup\{d \ge x | (f - h_x)(y) \ge 0 \text{ for } y \in [x, d]\}$. Note that, by continuity, $x < c < z$ and $(f - h_x)(c) = 0$; and, by construction, for any $d > c$ there exists $y \in (c, d)$ such that $(f - h_x)(y) < 0$. So, there exists a decreasing sequence y_n such that $y_n \to c$ and $(f - h_x)(y_n) < 0$. It follows that $(f - h_x)'(c+) = f'(c+) - f'(x+) \le 0$. This contradicts the assumption that $f'(\cdot)$ is strictly increasing. On the other hand, if $z < x$, let $c = \inf\{d \le x | (f - h_x)(y) \ge 0 \text{ for } y \in [d, x]\}$. Then $z < c < x$, $(f - h_x)(c) = 0$, and there exists a decreasing sequence y_n such that $y_n \to c$ and $(f - h_x)(y_n) \ge 0$. Therefore, $(f - h_x)'(c+) = f'(c+) - f'(x+) \ge 0$. This again contradicts the assumption that $f'(\cdot)$ is strictly increasing. So, we conclude $f(y) \ge h_x(y)$. As ϵ can be arbitrarily small, we must have $f(y) \ge g_x(y)$.

Now, suppose $f'(x+)$ is increasing, and let $h(x) = f(x) + \frac{\epsilon}{2}x^2$ for some $\epsilon > 0$. Then $h'(x+)$ is strictly increasing. By the first part of the proof, $h(y) = f(y) + \frac{\epsilon}{2}y^2 \geq h'(x+)(y-x) + h(x) = (f'(x+) + \epsilon x)(y-x) + f(x) + \frac{\epsilon}{2}x^2 = g_x(y) + \epsilon x(y-x) + \frac{\epsilon}{2}x^2$. Letting $\epsilon \to 0$ on both sides, we have $f(y) \geq g_x(y)$. The proof is complete. $\square$

Proposition 3.15. *Suppose $E \subset \mathbb{R}^n$ is convex, $f : E \to \mathbb{R}$ is continuous, the set of polyhedrons $E_1, \ldots, E_m$ is a partition of E, and f is convex on each of $E_1, \ldots, E_m$. Then f is convex if and only if $-f'(\boldsymbol{x}; -\boldsymbol{z}) \leq f'(\boldsymbol{x}; \boldsymbol{z})$ for $\boldsymbol{x} \in E$ for $\boldsymbol{z} \in \mathbb{R}^n$.*

Proof. The "only if" part is always true for a convex function on $\mathbb{R}^n$. For the "if part", note that $g : (a,b) \to \mathbb{R}$ is convex iff it is continuous with increasing left- or right-derivative. (For a proof, see Proposition 3.14 in the Appendix.) It follows that if $g : [a,b] \to \mathbb{R}$ is continuous and piecewise convex on $[a_0, a_1]$, $[a_1, a_2]$, ..., $[a_{m-1}, a_m]$, where $a = a_0 < \cdots < a_m = b$, then to show g is convex, we only need to show that $g'(x-) \leq g'(x+)$ for $x \in [a,b]$. To apply this argument to f, note that f is convex if $\phi(s) = f(\boldsymbol{y} + s\boldsymbol{z})$ is convex as a function of s for each $\boldsymbol{y} \in E$ and $\boldsymbol{z} \in \mathbb{R}^n$. As E is convex, the domain of $\phi(\cdot)$ is an interval $J \subset \mathbb{R}$. As s varies in J, $\boldsymbol{y} + s\boldsymbol{z}$ intersects with $E_1, \ldots, E_m$ for s in (possibly empty) intervals $J_1, \ldots, J_m$, respectively. As $E_1, \ldots, E_m$ forms a partition of E, $J_1, \ldots, J_m$ forms a partition of J. It follows that $\phi(s)$ is piecewise convex. Therefore, $\phi(s)$ is convex if $\phi'(s-) \leq \phi'(s+)$ for $s \in J$. Set $\boldsymbol{x} = \boldsymbol{y} + s\boldsymbol{z}$, then $\phi'(s-) = -f'(\boldsymbol{x}; -\boldsymbol{z})$ and $\phi'(s+) = f'(\boldsymbol{x}; \boldsymbol{z})$. It follows that $-f'(\boldsymbol{x}; -\boldsymbol{z}) \leq f'(\boldsymbol{x}; \boldsymbol{z})$ implies f is convex. $\square$

Proof of Lemma 3.8: We omit the subscript t since there will be no ambiguity.

It is easy to see that the continuity of $u(\cdot)$ follows from the Dominated Convergence Theorem, as $U_{d,p}(\boldsymbol{y}) \leq \beta \sum_i d_i + \rho \|v\|_u$, where $\|\cdot\|_u$ is the uniform norm. From our discussion, it remains to be shown that $-U'_{d,p}(\boldsymbol{y}; -\boldsymbol{z}) \leq U'_{d,p}(\boldsymbol{y}; \boldsymbol{z})$ and $U'_{d,p}(\boldsymbol{y}; \boldsymbol{z}) \leq \beta \sum_i |z_i|$ for $\boldsymbol{y} \in S^\circ$ for $\boldsymbol{z} \in \mathbb{R}^n$. To do this, for $\boldsymbol{y} \in \mathbb{R}^n$, we let $J^-(\boldsymbol{y}) = \{i | y_i < 0\}$, $J^0(\boldsymbol{y}) = \{i | y_i = 0\}$, and $J^+(\boldsymbol{y}) = \{i | y_i > 0\}$.

Suppose $\boldsymbol{y} \in S^\circ$ so that every direction $\boldsymbol{z} \in \mathbb{R}^n$ is feasible. We first derive $U'_{d,p}(\boldsymbol{y}; \boldsymbol{z})$ and $U'_{d,p}(\boldsymbol{y}; -\boldsymbol{z})$. Note that

$$L(\boldsymbol{y}, \boldsymbol{d}) = \beta \sum_{i \in J^-(y-d)} (d_i - y_i),$$

and

$$x_i(\boldsymbol{y}) = \begin{cases} (y_i - d_i) + p_i(N - \sum_{j \in J^+(y-d)} (y_j - d_j)) & \text{for } i \in J^+(\boldsymbol{y} - \boldsymbol{d}), \\ p_i(N - \sum_{j \in J^+(y-d)} (y_j - d_j)) & \text{for } i \in J^0(\boldsymbol{y} - \boldsymbol{d}) \cup J^-(\boldsymbol{y} - \boldsymbol{d}). \end{cases}$$

If t is small enough, then $J^-(\boldsymbol{y} + t\boldsymbol{z} - \boldsymbol{d}) = J^-(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^-(\boldsymbol{z}))$, $J^0(\boldsymbol{y} + t\boldsymbol{z} - \boldsymbol{d}) = J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^0(\boldsymbol{z})$, and $J^+(\boldsymbol{y} + t\boldsymbol{z} - \boldsymbol{d}) = J^+(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^+(\boldsymbol{z}))$. So for $\boldsymbol{y} + t\boldsymbol{z}$, we have

$$L(\boldsymbol{y} + t\boldsymbol{z}, \boldsymbol{d}) = \beta \sum_{i \in J^-(y-d) \cup (J^0(y-d) \cap J^-(z))} (d_i - y_i - tz_i),$$

and

$$x_i(\boldsymbol{y} + t\boldsymbol{z})$$
$$= \begin{cases} (y_i + tz_i - d_i) + p_i(N - \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^+(z))} (y_j + tz_j - d_j)) & i \in J^+(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^+(\boldsymbol{z})), \\ p_i(N - \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^+(z))} (y_j + tz_j - d_j)) & i \in J^-(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap (J^-(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))). \end{cases}$$

Similarly, if t is small enough, $J^-(\boldsymbol{y} - t\boldsymbol{z} - \boldsymbol{d}) = J^-(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^+(\boldsymbol{z}))$,

$J^0(\boldsymbol{y}-t\boldsymbol{z}-\boldsymbol{d}) = J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^0(\boldsymbol{z})$, and $J^+(\boldsymbol{y}-t\boldsymbol{z}-\boldsymbol{d}) = J^+(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^-(\boldsymbol{z}))$. So for $\boldsymbol{y}-t\boldsymbol{z}$, we have

$$L(\boldsymbol{y}-t\boldsymbol{z},\boldsymbol{d}) = \beta \sum_{i\in J^-(y-d)\cup(J^0(y-d)\cap J^+(z))} (d_i - y_i + tz_i),$$

and

$$x_i(\boldsymbol{y}-t\boldsymbol{z}) = \begin{cases} (y_i - tz_i - d_i) + p_i(N - \sum_{j\in J^+(y-d)\cup(J^0(y-d)\cap J^-(z))}(y_j - tz_j - d_j)) & i \in J^+(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^-(\boldsymbol{z})), \\ p_i(N - \sum_{j\in J^+(y-d)\cup(J^0(y-d)\cap J^-(z))}(y_j - tz_j - d_j)) & i \in J^-(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap (J^+(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))). \end{cases}$$

It follows that

$$L(\boldsymbol{y}+t\boldsymbol{z},\boldsymbol{d}) - L(\boldsymbol{y},\boldsymbol{d}) = -\beta t \sum_{i\in J^-(y-d)\cup(J^0(y-d)\cap J^-(z))} z_i,$$

and

$$L(\boldsymbol{y}-t\boldsymbol{z},\boldsymbol{d}) - L(\boldsymbol{y},\boldsymbol{d}) = \beta t \sum_{i\in J^-(y-d)\cup(J^0(y-d)\cap J^+(z))} z_i.$$

We also have

$$x_i(\boldsymbol{y}+t\boldsymbol{z}) - x_i(\boldsymbol{y}) = \begin{cases} tz_i - p_i t \sum_{j\in J^+(y-d)\cup(J^0(y-d)\cap J^+(z))} z_j & i \in J^+(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^+(\boldsymbol{z})), \\ -p_i t \sum_{j\in J^+(y-d)\cup(J^0(y-d)\cap J^+(z))} z_j & i \in J^-(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap (J^-(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))), \end{cases}$$

and

$$x_i(\boldsymbol{y} - t\boldsymbol{z}) - x_i(\boldsymbol{y})$$
$$= \begin{cases} -tz_i + p_i t \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^-(z))} z_j & i \in J^+(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^-(\boldsymbol{z})), \\ p_i t \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^-(z))} z_j & i \in J^-(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap (J^+(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))). \end{cases}$$

Set

$$\boldsymbol{w}^+ = \frac{\boldsymbol{x}(\boldsymbol{y} + t\boldsymbol{z}) - \boldsymbol{x}(\boldsymbol{y})}{t}, \quad \boldsymbol{w}^- = \frac{\boldsymbol{x}(\boldsymbol{y} - t\boldsymbol{z}) - \boldsymbol{x}(\boldsymbol{y})}{t}.$$

Then

$$(v \circ \boldsymbol{x})'(\boldsymbol{y}; \boldsymbol{z}) = v'(\boldsymbol{x}; \boldsymbol{w}^+), \quad (v \circ \boldsymbol{x})'(\boldsymbol{y}; -\boldsymbol{z}) = v'(\boldsymbol{x}; \boldsymbol{w}^-).$$

It follows that

$$U'_{d,p}(\boldsymbol{y}; \boldsymbol{z}) = -\beta \sum_{i \in J^-(y-d) \cup (J^0(y-d) \cap J^-(z))} z_i + \rho v'(\boldsymbol{x}; \boldsymbol{w}^+),$$

and

$$U'_{d,p}(\boldsymbol{y}; -\boldsymbol{z}) = \beta \sum_{i \in J^-(y-d) \cup (J^0(y-d) \cap J^+(z))} z_i + \rho v'(\boldsymbol{x}; \boldsymbol{w}^-),$$

where

$$w_i^+ = \begin{cases} z_i - p_i \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^+(z))} z_j & i \in J^+(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^+(\boldsymbol{z})), \\ -p_i \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^+(z))} z_j & i \in J^-(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap (J^-(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))), \end{cases}$$

and

$$w_i^- = \begin{cases} -z_i + p_i \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^-(z))} z_j & i \in J^+(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap J^-(\boldsymbol{z})), \\ p_i \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^-(z))} z_j & i \in J^-(\boldsymbol{y} - \boldsymbol{d}) \cup (J^0(\boldsymbol{y} - \boldsymbol{d}) \cap (J^+(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))). \end{cases}$$

To show $U_{d,p}(\cdot)$ is convex, let $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ be such that

$$v_i^+ = \begin{cases} z_i \sum_j p_j + p_i \sum_{j\in(J^0(y-d)\cap J^-(z))} z_j - p_i \sum_{j\in(J^0(y-d)\cap J^+(z))} z_j & i \in J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^+(\boldsymbol{z}), \\ -z_i \sum_j p_j + p_i \sum_{j\in(J^0(y-d)\cap J^-(z))} z_j - p_i \sum_{j\in(J^0(y-d)\cap J^+(z))} z_j & i \in J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^-(\boldsymbol{z}), \\ p_i \sum_{j\in(J^0(y-d)\cap J^-(z))} z_j - p_i \sum_{j\in(J^0(y-d)\cap J^+(z))} z_j & i \text{ in elsewhere,} \end{cases}$$

and

$$v_i^- = \begin{cases} z_i(1-\sum_j p_j) & \text{for } i \in J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^+(\boldsymbol{z}), \\ -z_i(1-\sum_j p_j) & \text{for } i \in J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^-(\boldsymbol{z}), \\ 0 & \text{for } i \text{ in elsewhere.} \end{cases}$$

It is easy to see that $\boldsymbol{v}^+ + \boldsymbol{v}^- = \boldsymbol{w}^+ + \boldsymbol{w}^-$ and $\boldsymbol{v}^+ \in \mathcal{H}$, where $\mathcal{H}$ is defined by (3.21). So, by Corollary 3.6, we have

$$\begin{aligned} U'_{d,p}(\boldsymbol{y};\boldsymbol{z}) + U'_{d,p}(\boldsymbol{y};-\boldsymbol{z}) &= \beta \sum_{J^0(y-d)} |z_i| + \rho(v'(\boldsymbol{x};\boldsymbol{w}^+) + v'(\boldsymbol{x};\boldsymbol{w}^-)) \\ &\geq \beta \sum_{J^0(y-d)} |z_i| + \rho v'(\boldsymbol{x};\boldsymbol{v}^+ + \boldsymbol{v}^-) \\ &\geq \beta \sum_{J^0(y-d)} |z_i| - \rho\beta(\frac{1}{2}\sum_i |\boldsymbol{v}_i^+| + \sum_i |\boldsymbol{v}_i^-|) \\ &\geq \beta \sum_{J^0(y-d)} |z_i| - \rho\beta(\sum_j p_j \sum_{i\in J^0(y-d)} |z_i| + (1-\sum_j p_j) \sum_{i\in J^0(y-d)} |z_i|) \\ &\geq (1-\rho)\beta \sum_{J^0(y-d)} |z_i|. \end{aligned}$$

Therefore, $U_{d,p}(\cdot)$ is convex on S°. Since S is locally simplicial, the continuous extension of $U_{d,p}(\cdot)$ from S° to S must be convex. (See for example Rockafellar [1970] Theorem 10.3.) Therefore, $u(\cdot)$ is convex.

To show that $U'_{d,p}(\boldsymbol{y};\boldsymbol{z}) \leq \beta \sum_i |z_i|$ for $\boldsymbol{y} \in S^\circ$ for $\boldsymbol{z} \in \mathbb{R}^n$, let $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ be such that

$$v_i^+ = \begin{cases} z_i \sum_j p_j - p_i \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^+(z))} z_j & i \in J^+(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^+(\boldsymbol{z})), \\ -p_i \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^+(z))} z_j & i \in J^-(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap (J^-(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))), \end{cases}$$

and

$$v_i^- = \begin{cases} z_i(1 - \sum_j p_j) & \text{if} \quad i \in J^+(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap J^+(\boldsymbol{z})), \\ 0 & \text{if} \quad i \in J^-(\boldsymbol{y}-\boldsymbol{d}) \cup (J^0(\boldsymbol{y}-\boldsymbol{d}) \cap (J^-(\boldsymbol{z}) \cup J^0(\boldsymbol{z}))). \end{cases}$$

Then $\boldsymbol{v}^+ + \boldsymbol{v}^- = \boldsymbol{w}^+$ and $\boldsymbol{v}^+ \in \mathcal{H}$. It follows from Corollary 3.6 that

$$\begin{aligned} U'_{d,p}(\boldsymbol{y}; \boldsymbol{z}) &= -\beta \sum_{i \in J^-(y-d) \cup (J^0(y-d) \cap J^-(z))} z_i + \rho v'(\boldsymbol{x}; \boldsymbol{w}^+) \\ &\leq \beta \sum_{i \in J^-(y-d) \cup (J^0(y-d) \cap J^-(z))} |z_i| + \rho(v'(\boldsymbol{x}; \boldsymbol{v}^+) + v'(\boldsymbol{x}; \boldsymbol{v}^-)) \\ &\leq \beta \sum_{i \in J^-(y-d) \cup (J^0(y-d) \cap J^-(z))} |z_i| + \rho\beta(\frac{1}{2} \sum_i |v_i^+| + \sum_i |v_i^-|) \\ &\leq \beta \sum_{i \in J^-(y-d) \cup (J^0(y-d) \cap J^-(z))} |z_i| + \rho\beta((\sum_j p_j + (1 - \sum_j p_j)) \sum_{j \in J^+(y-d) \cup (J^0(y-d) \cap J^+(z))} |z_i|) \\ &\leq \beta \sum_i |z_i| \end{aligned}$$

So, $U'_{d,p}(\boldsymbol{y}; \boldsymbol{z}) \leq \beta \sum_i |z_i|$ holds for each $\boldsymbol{y} \in S^\circ$ and $\boldsymbol{z} \in \mathbb{R}^n$. By Lemma 3.12 in the Appendix, this implies $U_{d,p}(\cdot)$ is Lipschitz with a Lipschitz constant $n\beta$ for all $(\boldsymbol{d}, \boldsymbol{p})$. It follows that $u'(\boldsymbol{y}; \boldsymbol{z}) = \int U'_{d,p}(\boldsymbol{y}; \boldsymbol{z})\, d\mu \leq \beta \sum_i |z_i|$. We are done. $\square$

Chapter 4

Concluding Comments and Future Research

In the first part of the thesis, we described an equilibrium model of P2P product sharing or collaborative consumption. We characterized equilibrium outcomes, including ownership and usage levels, consumer surplus, and social welfare. We compared each outcome in systems with and without collaborative consumption and examined the impact of various problem parameters including rental price, platform's commission rate, cost of ownership, owner's extra wear and tear cost, and renter's inconvenience cost. Our findings indicate that collaborative consumption can result in either higher or lower ownership and usage levels, with higher ownership and usage levels more likely when the cost of ownership is high. We showed that consumers always benefit from collaborative consumption, with individuals who, in the absence of collaborative consumption, are indifferent between owning and not owning benefitting the most. We studied both profit maximizing and social welfare maximizing platforms and compared equilibrium outcomes under both in terms of ownership, usage, and social welfare. We found that a not-for-profit platform would always charge a lower price and, therefore, lead to lower ownership and usage than a for-profit platform (suggesting that a regulator may be able to nudge a for-profit platform toward outcomes with higher social welfare by putting a cap on rental price). We also showed that the platform's profit is not monotonic in the cost of ownership, implying that a platform is least profitable when the cost of ownership is either very high or very low (suggesting that a platform may have an incentive to affect the cost of ownership by, for example, imposing membership fees or providing subsidies). In addition, we observed that platform profit can be non-monotonic in the extra wear and tear cost, suggesting that a for-profit platform may not have an incentive to eliminate this cost.

We described extensions of our analysis to several settings. In each case, we confirmed the robustness of our main results, but also uncovered additional insights. For example, for systems with a third party service provider, collaborative consumption is more likely to lead to more ownership when the service level of the third party service provider is higher. In settings where the platform may own products, the platform would profit from these products only if the cost of ownership and commission fee are sufficiently high. For the case where individuals are heterogeneous in their sensitivity to extra wear and tear and inconvenience (and similarly for the case where usage is endogenous), the impact of price on ownership and usage is no longer monotonic. In settings where individuals with higher usage are more prevalent, collaborative consumption is more likely to lead to less ownership than in settings where individuals with lower usage are more prevalent.

In the second part of the thesis, we considered the inventory repositioning problem in B2C product sharing networks, where demand, rental periods, and return locations are stochastic. We formulated the problem as a Markov decision process and showed that the problem to be solved in each period is one that involves solving a convex optimization problem. We proved that the optimal policy is specified in terms of a region in the state space, inside of which it is optimal not to carry out any repositioning and outside of which it is optimal to reposition inventory. We also proved that, in repositioning, it is always optimal to do so such that the system moves to a new state that is on the boundary of the no-repositioning region and provided a simple check for when a state is in the no-repositioning region. We showed that, unlike many classical inventory problems, the optimal inventory repositioning policy cannot be characterized by simple thresholds in the state space. Moreover, the characterization of the no-repositioning region and of the optimal cost function can be leveraged to improve the efficiency at which the optimal policy can be computed.

Our work takes a first step toward the study of the impact of P2P product sharing as well as the inventory management for B2C product sharing. It has the potential to open up new directions for future research in the sharing and on-demand economy. We mention a few examples. It would be interesting to consider a setting where there is competition between a P2P platform and a B2C service provider, with renters having the options of using one or both types of services. For cars, such a scenerio has already unfolded in many places, where P2P platforms such as Turo and Getaround operate alongside B2C service providers such as Zipcar and Car2go. However, it is unclear whether both types of services could co-exist or either would prevail in the long run. A P2P platform could afford more competitive pricing than a B2C service provider (because owners derive positive utility from usage, they are willing to rent out their products at a lower price). On the other hand, a B2C service provider may mitigate this disadvantage if it could benefit from economies of scale. Given the fact that B2C service providers have full control over their products, they could also more easily improve product utilization and rental experience than a P2P platform. This could give rise to a situation where under some conditions P2P rental dominates the market while under other conditions B2C rental dominates. This could also lead to a situation where the two types of services serve different market segments.

It would also be interesting to consider a setting where there is a duopoly competition between two P2P product sharing platforms with multi-homing owners and renters (i.e., they can participate in one or both of the platforms). Given that the effective demand the platform would face is non-monotonic in price, competition may not necessarily lead to lower prices. This is also likely to have interesting implications for product ownership, usage, consumer surplus and social welfare.

A related setting is where a decision maker can decide to be either a P2P platform, a B2C service provider, or a hybrid of the two. This would provide an answer to the question of whether B2C service providers have an incentive in investing in a P2P platform. A B2C service provider could benefit from running a P2P platform if the introduction of the platform could increase rental demand while decreasing its spending on products. However, introducing P2P product sharing may also invite competition to its core business, which could negatively impact its overall profit. Therefore, there could be conditions under which a B2C service provider may or may not want to engage in peer-to-peer product sharing.

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